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Inverse Semigroups on the 2-Cell.

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ABSTRACT

In this paper we investigate the structure of inverse semigroups on the 2-cell with an identity. Let S denote such a semigroup. After the preliminaries in chapter I, it is shown that the minimal ideal, K, is a point and hence S has a zero.

In chapter II, the case when E has a cut point is considered. If $e \in E$ is a cut point, then $H(e)$ must be a circle group, SeS is the union of $H(e)$ with the bounded component of $R^2 \setminus H(e)$, and $E \cap SeS$ is a min thread. Then a characterization of semigroups of this type is established by showing that if E has a cut point, then S must be a min cone over a circle group, and E must be a min thread.

In chapter III, considering the case when E has no cut point, it is shown that the inverse homeomorphism which maps s onto s^{-1}, is the identity map and $s = s^{-1}$ for all $s \in S$. Hence it follows that S is commutative and a union of groups. The maximal group $H(1)$ then must be either $Z_2 \times Z_2$, Z_2, or $\{1\}$. In the two cases when $H(1) \neq \{1\}$, $0 \in \overline{S \setminus E}$, and if $H(1) \approx Z_2 \times Z_2$, then $0 \in \operatorname{int} S$. As a partial characterization of semigroups of this

type, it is shown that if E has no cut point and $H(1) \approx Z_2 \times Z_2$, then S is the continuous monotone homomorphic image of the cartesian product of the min cone over Z_2 with itself, if and only if $M(e)$ is connected for each $e \in E$. Using the min conver over Z_2 and the min cone over the two element semilattice as building blocks, examples are constructed to illustrate the existence of semigroups for the various cases.

Chapter IV considers only algebraic inverse semigroups. It is shown that for each $a \in S$, the row and column idempotents of the powers of a generate a lattice denoted by Λ_a. The following three conditions are considered for $a \mathcal{H} b$:

1) $(a,b) \in \mu$, the maximum idempotent-separating congruence;
2) $\Lambda_a = \Lambda_b$;
3) $a^n a^{-n} = b^n b^{-n}$ where $a^n a^{-n}$ and $b^n b^{-n}$ are the row idempotents for a^n and b^n, respectively.

It is shown that 1) implies 2) and 2) implies 3), and examples are given to show that the two implications are not reversible.

INTRODUCTION

One of the questions which arises in the study of topological semigroups is "given a topological space, can the semigroups defined on that space be classified?" The arc was considered first, and after semigroups on an arc were characterized, the next space to be considered was, naturally, the 2-cell. Various types of semigroups on the 2-cell have been studied and classified. Among them are the affine semigroups, the semigroups with group boundary, semilattices whose "upper sets" are connected, semigroups with trivial multiplication on the boundary, and semigroups with the boundary a union of threads.

In this paper we shall consider the problem of characterizing inverse semigroups with an identity on the 2-cell. For semigroups of this type, the set of idempotents E is a retract, and we approach the problem by considering the following two cases: 1) when E has a cut point, and 2) when E does not have a cut point and is a 2-cell. In chapter II, we consider case one, with the result being that the min cone over the circle group is the only semigroup of this type.

The study of case two in chapter III was not as fruitful and a complete characterization was not established.

However, these important results were obtained: the semigroup must be commutative, a union of groups, and the maximal group containing 1 must be either $Z_2 \times Z_2$, Z_2 , or $\{1\}$. The homeomorphisms, defined by mapping x onto x^{-1} and x onto ax for fixed $a \in H(1)$, were used in establishing these facts. Chapter III concludes the study of Topological inverse semigroups on the 2-cell with several examples.

In chapter IV, we consider general algebraic inverse semigroups and a lattice of idempotents is shown to exist for each element. An example is given to show that if two elements have the same lattice, they do not necessarily have to be related modulo the maximum idempotent-separating congruence.

CHAPTER I

<u>Introduction</u>. The purpose of this chapter is to establish notation, to introduce terminology and definitions, and to state previous results in the topology of the plane and in semigroups, which will be used and referred to in later chapters. For a more detailed and complete background, the suggested references are Kelley [11] and Whyburn [20] for topology, Clifford and Preston [5] and [6] for algebraic semigroups, and Hofmann and Mostert [8] for topological semigroups.

<u>Topological Preliminaries</u>. Throughout this paper, R^2 will denote the Cartesian plane with the usual topology. If A and B are subsets of R^2, then $A \backslash B$ is understood to mean the complement of B in A. The closure of a subset A is indicated by $\overline{A}$ and the empty set is denoted by $\square$. The boundary of A, written $\mathrm{Bd}A$, is $A \cap \overline{R^2 \backslash A}$; it will always mean the boundary with respect to R^2. If A is a simple closed curve, then $\mathrm{int}\,A$ represents the bounded component of $R^2 \backslash A$ and $\mathrm{ext}\,A$ denotes the unbounded component of $R^2 \backslash A$.

The Alexander-Kolmogoroff-Spanier-Wallace cohomology theory is used with the integers as the coefficient group.

For $A \subset R^2$, A is said to be acyclic if A has the cohomology groups of a point.

Because of their importance and frequent use in establishing results in this paper, the following well known theorems are presented without proofs. The statements are modified and restricted for clarity in the special cases considered here.

<u>Theorem 1.1</u> [20]. If T is an arc from a to b in a 2-cell S such that $\{a,b\} \subset \mathrm{BdS}$ and $T \cap \mathrm{int}\, S \neq \Box$, then T separates S.

<u>Theorem 1.2</u> [20]. If x and y are elements of $(\mathrm{BdS}) \setminus T$ and the two points a and b separate x from y in BdS, then T separates x from y in S.

<u>Theorem 1.3</u> [20]. If $T \cap \mathrm{BdS} = \{a,b\}$, then T separates S, and $S \setminus T$ has exactly two components.

<u>Theorem 1.4</u> [20]. If T is an arc spanning a 2-cell S from a to b, R_1 and R_2 are the two components of $S \setminus T$, and E is a 2-cell contained in S such that a subarc T_1 of T is contained in BdE, then if $t \in T_1$, there exists open set U, U_1, and U_2 and an arc T_2 such that $t \in U$, $U = U_1 \cup U_2 \cup T_2$, $U \cap R_1 = U_1$, $U \cap R_2 = U_2$, $U \cap T_1 = T_2$ and either $U_1 \subset E$ or

$U_2 \subset E$.

Theorem 1.5 [20]. If T is an arc spanning a 2-cell S from a to b and E is a 2-cell contained in S such that $T \subset \mathrm{Bd}E$, then E is contained in the closure of one and only one of the two components of $S \setminus T$.

Theorem 1.6 [3]. A subset E of a 2-cell S is a retract of S if and only if E is a locally connected continuum which does not separate the plane.

Theorem 1.7 [21]. A retract E of a 2-cell S is a 2-cell if it is cyclicly connected, that is, if E has no cut point.

Semigroup Preliminaries. A semigroup S is a non-empty set together with an associative binary operation called multiplication. A topological semigroup S is a semigroup which is a Hausdorff topological space such that the multiplication is continuous. An element e of a semigroup S is called idempotent if $e^2 = e$. Throughout the remainder of this chapter, S will denote a topological semigroup and E will denote the set of idempotents of S. If $1 \in S$, such that $1 \cdot a = a \cdot 1 = a$ for all $a \in S$, then 1 is called an identity for S. If $0 \in S$, such that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$, then 0 is called a zero for S.

By a <u>left</u> [<u>right</u>] <u>ideal</u> of S we mean a non-empty subset of S such that $SA \subseteq A$ [$AS \subseteq A$]. A is an <u>ideal</u> if A is both a right and left ideal. A compact semigroup S contains a unique <u>minimal ideal K</u>, called the <u>kernel</u> of S.

A continuous function f from S into a topological semigroup T is called a <u>homomorphism</u> if $f(ab) = f(a)f(b)$. If, in addition, f is a homeomorphism, then f is called an <u>iseomorphism</u>.

If $a \in aSa$ for each $a \in S$, then S is called a <u>regular</u> semigroup. For $a,b \in S$, if $aba = a$ and $bab = b$, then b is called an <u>inverse</u> of a. An <u>inverse semigroup</u> is a semigroup in which every element has a unique inverse. For $a \in S$ denote the unique inverse of a by a^{-1}. A semigroup is called <u>Clifford</u> if it is a union of groups.

<u>Theorem 1.8</u> [5]. The following are equivalent:

i) S is regular, and any two idempotents of S commute;

ii) every principal right ideal and every principal left ideal of S has a unique idempotent generator;

iii) S is an inverse semigroup.

<u>Theorem 1.9</u> [5]. A semigroup S is an inverse semigroup if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class of

S contains exactly one idempotent.

<u>Theorem 1.10</u> [5]. For any elements a, b of an inverse semigroups S, we have

$$(a^{-1})^{-1} = a \quad \text{and} \quad (ab)^{-1} = b^{-1}a^{-1}.$$

Certain equivalence relations, called Green's relations, play an important role in the theory of semigroups. Four of these relations for a compact inverse semigroup are defined as follows:

$$\mathcal{L} = \{(a,b) : Sa = Sb\},$$
$$\mathcal{R} = \{(a,b) : aS = bS\},$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$
$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} = \{(a,b) : SaS = SbS\}.$$

$H(a)$, $L(a)$, $R(a)$ and $D(a)$ will denote the corresponding equivalence classes containing a. If $e \in E$, then $H(e)$ is the maximal group containing e.

A semigroup is <u>stable</u> if $Sa \subset Sab$ implies $Sa = Sab$ and $xS \subset yxS$ implies $xS = yxS$ for each $a, b, x, y \in S$.

A <u>thread</u> is a semigroup on a closed real interval in which the endpoints act as zero and identity. A <u>min thread</u> is a thread whose multiplication is defined by $xy = \min\{x, y\}$ where the minimum is taken with respect to the usual ordering of the reals. I will denote the min thread defined on the closed real interval from 0 to 1. Let S and D be compact semigroups with kernels K_S and K_D respectively. Define the congruence relation C on $S \times D$ by:

If $y \in K_D$, then $(x,y)C(s,t)$ if, and only if $y = t$.

If $y \notin K_D$, then $(x,y)C(s,t)$ if, and only if $(x,y)=(s,t)$.

C is a closed congruence and the semigroup $\frac{(S \times D)}{C}$ is called the D cone over S [1]. If I or any min thread is chosen as D, then $\frac{(S \times I)}{C}$ is called the min con over S. Of particular interest in this paper is the min cone over the circle group, the min cone over Z_2, and the min cone over the 2 element semilattice, all of which are inverse semigroups.

We now further restrict our attention by letting S denote only an inverse semigroup on the 2-cell with an identity 1. There is a natural partial ordering on S defined by $a \leq b$ if $a = ab^{-1}a$.

Theorem 1.11 [13]. The following are equivalent:

(1) $x \leq y$ (2) $xx^{-1} = y^{-1}x$ (3) $x^{-1}x = x^{-1}y$

(4) $xx^{-1} = xy^{-1}$ (5) $x = xx^{-1}y$ (6) $yx^{-1}x = x$ and

(7) $x^{-1} \leq y^{-1}$.

Theorem 1.12 [13]. S is stable, and if $x \leq y$ and $x\mathcal{D}y$, then $x = y$.

Theorem 1.13 [8]. If D is a $\mathcal{D}$-class of S, then $\dim D \leq 1$.

Theorem 1.14. The kernel K of S is the one point group, so S has a zero.

Proof: The existence of K in compact semigroups is well known as is the fact that K is connected and consists of one $\mathcal{D}$-class. If e and f are idempotents of K, then $ef \leq e$ and $ef \leq f$, but $ef\mathcal{D}f\mathcal{D}e$, so $e = ef = f$ by stability. Hence K contains one idempotent and K is a group. Now $0 = H^1(S) \approx H^1(K)$, so $\dim K = 0$, and since K is connected, K consists of a single point.

Remark: It will be assumed that S has a zero without reference to this theorem throughout the remainder of the paper.

Theorem 1.15 [2]. If $e \in E$, then either $\dim H(e) = 0$ or $H(e)$ is a circle group and $\dim H(e) = 1$.

Theorem 1.16 [15]. There exists a min thread $T \subset E$ from 0 to 1.

Theorem 1.17. If $\mathrm{Bd}S = H(1)$, a circle group, then for each $x \in S$, either $xH(1)$ is homeomorphic to a circle and $N_x = \{h \in H(1) : xh = x\}$ is finite and cyclic, or $xH(1) = x$ and $N_x = H(1)$.

Proof: This theorem follows from the well-known fact that the closed subgroups of the circle group are finite cyclic or the group itself.

Theorem 1.18. If $e \in E$ and $H(e)$ is a circle group, then $SeS = H(e) \cup \text{int } H(e)$ and $0 \in \text{int } H(e)$.

Proof: Let T be a min thread from 0 to 1, and consider the map H from $T \times SeS$ onto SeS defined by $H(t, ses_1) = tses_1$. Clearly H is continuous and $H(1, ses_1) = ses_1$ and $H(0, ses_1) = 0$ for all $ses_1 \in SeS_0$. The Generalized Homotopy Theorem [18] implies $H^n(SeS) \approx H^n(0)$ for all n, hence SeS is acyclic. By [2], $0 \in \text{int } H(e)$, and clearly $H(e) \subset SeS$. Therefore, $H(e) \cup \text{int } H(e) \subset SeS$.

If $a \in SeS \cap \text{ext } H(e)$, then aT is an arc from a to 0, so there exists $t \in T$ such that $at \in H(e)$ and $a^{-1}at = e$. We may assume $1 \neq e$ since the theorem is obviously true if $e = 1$ and $1 \notin \text{int } H(e)$ since $H(1)$ is contained in the boundary of S [18]. Therefore, $e \in T$ and e and t compare. If $e \leq t$, then $e = e \cdot e = a^{-1}at \cdot e = a^{-1}ae$ and $e \leq a^{-1}a$. If $t \leq e$, then $t = et = a^{-1}at \cdot t = a^{-1}at = e$ and again $e \leq a^{-1}a$.

Therefore,

$$SaS \subset SeS \subset Sa^{-1}aS = SaS \ ,$$

hence $SeS = Sa^{-1}aS$ and $e \mathcal{J} a^{-1}a$. By stability, $e = a^{-1}a$ and similarly $e = aa^{-1}$, hence $a \in H(e)$, a contradiction. Therefore, $SeS = H(e) \cup \text{int } H(e)$.

Remark: The theorems and results of this chapter will be used throughout the remainder of this paper without specific reference.

A note of warning. Many of the definitions and theorems stated in this chapter only apply to inverse semigroups, and in particular, some only apply to inverse semigroups on the 2-cell with an identity. Hence, for the more general theory, one should consult the references mentioned in the introduction.

CHAPTER II

Introduction. The purpose of this chapter is to characterize inverse semigroups with an identity on the 2-cell whose set of idempotents E has a cut point. Using Dimension Theory, it is shown that for a cut point e of E, $H(e)$, the maximal group associated with e, is a circle group and further that if $f \leq e$ then $H(f)$ is also a circle group.

By considering the two cases $\dim E = 1$ and $\dim E = 2$, it is established that E must be a min thread and that the only inverse semigroup with an identity on the 2-cell whose set of idempotents E has a cut point is a min cone over a circle.

Preliminaries. Because of their importance in establishing this characterization, the following four theorems from Dimension Theory [8] will be stated without proofs. The statements will be modified and restricted to the special cases of this chapter.

Theorem 2.1 [8,pp.48]. The 2-cell cannot be disconnected by a subset of dimension ≤ 0.

Theorem 2.2 [8,pp.34]. If B is either

0-dimensional or one-dimensional and A is compact then

$$\dim (A \times B) = \dim A + \dim B.$$

Theorem 2.3 [8,pp.91] If f is a mapping of a compact space X onto a space Y and

$$\dim X - \dim Y = 1,$$

then there is a point of Y whose inverse image has dimension ≥ 1.

Theorem 2.4 [8,pp.46] Let U be an open set in R^2 which is neither empty nor dense in the space and let B be the boundary of U ; then $\dim B = 1$.

Results.

Theorem 2.5. Let S be an inverse semigroup with an identity on the 2-cell. If e is a cut point of E, then $H(e)$ is a circle group.

Proof: Let f be the retraction of S onto E defined by $r(x) = ss^{-1}$ [11]. Let $E_1 \cup E_2 = E \setminus e$ be a separation of E by e. Clearly $r^{-1}(E_1) \cap r^{-1}(E_2) = \square$, so suppose $x \in \overline{r^{-1}(E_1)} \cap r^{-1}(E_2)$. Then there is a sequence $\{x_n\}$ contained in $r^{-1}(E_1)$ converging to x. Inversion is continuous [8], so $\{x_n^{-1}\}$ converges to x^{-1} and it follows that $\{x_n x_n^{-1}\}$ converges to xx^{-1}. But $x_n x_n^{-1} \in E_1$ for all n and $xx^{-1} \in E_2$, a contradiction since $\overline{E_1} \cap E_2 = \square$. Thus $r^{-1}(e)$ separates S and by Theorem 2.1, $\dim r^{-1}(e) \geq 1$. Clearly $R(e) \subset r^{-1}(e)$ and for $x \in r^{-1}(e)$; $xx^{-1} = e$ and $x = xx^{-1}x = ex$. It follows that $x \in R(e)$, so $R(e) = r^{-1}(e)$ and $\dim R(e) \geq 1$.

By Theorem 1., $\dim D(e) \leq 1$ and since $R(e) \subset D(e)$, $\dim R(e) \leq 1$. Thus $\dim R(e) = 1$ and hence $\dim D(e) = 1$. Inversion restricted to $R(e)$ is a homeomorphism onto $L(e)$ and dimension is topologically invariant, so $\dim L(e) = 1$. $L(e)$ is compact, so by Theorem 2.2

$$\dim (L(e) \times R(e)) = \dim L(e) + \dim R(e) = 2.$$

Let m be the multiplication map from $L(e) \times R(e)$ onto $D(e)$ defined by $m(\ell, r) = \ell r$. By Theorem 2.3, there is an element ℓr of $D(e)$ such that $m^{-1}(\ell r)$ has dimension ≥ 1 and $m(\ell, r) = \ell r$. Now for $\ell r = \ell_1 r_1$,

$\ell S = \ell e S = \ell r S = \ell_1 r_1 S = \ell_1 e S = \ell_1 S$ and

$Sr = ser = S\ell r = S\ell_1 r_1 = Ser_1 = Sr_1$, so $\ell \mathcal{R} \ell_1$, $r \mathcal{L} r_1$ and $m^{-1}(\ell r) = H(\ell) \times H(r)$, hence $\dim (H(\ell) \times H(r)) \geq 1$.

By Green's translational lemmas, $H(e)$ is homeomorphic to $H(\ell)$ and $H(r)$, so $H(\ell) \times H(r)$ is homeomorphic to $H(e) \times H(e)$. Hence $\dim (H(e) \times H(e)) \geq 1$.

If $\dim H(e) = 0$, then by Theorem 2.2 $\dim (H(e) \times H(e)) = \dim H(e) + \dim H(e) = 0$, so by Theorem 1.15, $\dim H(e) = 1$ and $H(e)$ is a circle group.

Note that by this theorem 0 is not a cut point of E.

Corollary 2.6. If e is a cut point of E then $E \cap SeS$ is a min thread.

Proof: There exists a min thread $T \subset E$ from 1 to 0, [6], which clearly contains e and if $f \in T$ such that $f < e$, then $f \in SeS$. Clearly $T_1 = T \cap SeS \subset E \cap SeS$. $H(e)$ is a circle group, so for $x \in H(e) \setminus \{e\}$, let U be an open connected set about e and V and open connected set about x, such that $U \cap V = \Box$ and $Ux \subset V$. Let $f \in T_1 \cap U$ such that $f < e$. Then $fx \in V$, $fx \neq f$ and $fH(e)$ is a connected set containing at least two distinct points f and fx, so $\dim fH(e) \geq 1$. For all $x \in H(e)$, $fxS = feS = fS$, so $fH(e) \subset R(f)$ and $\dim R(f) \geq 1$. By the same argument as in the proof of the theorem $H(f)$ is a circle group.

Let $f = \inf A$ where $A = \{t \in T_1 \colon H(t)$ is a circle group and if $t < t_1$ for $t_1 \in T_1$ then $H(t_1)$ is a circle group$\}$. By the above argument $H(f)$ is not a circle group and $\dim H(f) = 0$. Let $B = \bigcap_{t \in A} StS$. Clearly B is compact, connected and contains f. Let $x \in BdB$. Then there is a sequence $\{x_n\}$ converging to x such that $x_n \in H(f_n)$, $f_{n+1} < f_n$ for all n , and $\{f_n\}$ converges to f. Inversion is continuous so $\{x_n^{-1}\}$ converges to x^{-1}. Now $\{x_n x_n^{-1}\}$ converges to xx^{-1} and $\{x_n^{-1}x_n\}$ converges to $x^{-1}x$; but $x_n x_n^{-1} = f_n = x_n^{-1}x_n$, so $xx^{-1} = f = x^{-1}x$ and $x \in H(f)$. Hence $\mathrm{BdB} \subset H(f)$, $f \in \mathrm{BdB}$ and $\dim \mathrm{BdB} = 0$. If $f \neq 0$ then $0 \notin \mathrm{BdB}$, so $\mathrm{int}\, B \neq \square$ and by Theorem 2.4 $\dim \mathrm{BdB} = 1$, a contradiction.

Therefore $f = 0 = \mathrm{BdB} = B = \bigcap_{t \in T} StS$ and clearly $T_1 = E \cap SeS$.

Theorem 2.7. If $\dim E = 1$, then E is a min thread and S is a min cone over a circle.

Proof: E is a tree, so the unique arc from 1 to 0 is a min thread T where every point of $T\setminus\{0,1\}$ is a cut point of E and hence whose $\mathcal{H}$-class is a circle group.

Suppose $e \in E\setminus T$. Then e lies on an arc A of idempotents for e to 0. Since T is closed, there exists $f \in A$ such that $f \notin T$ and f is a cut point of E. Hence $H(f)$ is a circle group, containing 0 in its interior, so $H(f) \cap T \neq \square$, a contradiction. Therefore $E=T$.

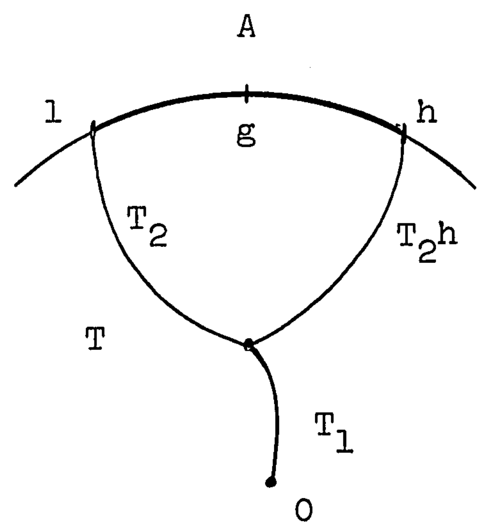
Let $e \in T\setminus\{0,1\}$. Then $SeS \subseteq SfS = H(f) \cup \mathrm{int}\, H(f)$ for $e < f$. But $H(e) \cap H(f) = \square$, so $H(e) \subset \mathrm{int}\, H(f) \subset S$, the 2-cell. Thus int $H(f) \subset$ int BdS and it follows that $H(e) \cap \mathrm{BdS} = \square$.

Clearly $H(1) \subset \mathrm{BdS}$, so let $x \in \mathrm{BdS}$. Then $SxS = SfS$ for some $f \in E$. Now $f \notin T \setminus \{0,1\}$ since $SeS = H(e) \cup \text{int } H(e) \subset \text{int BdS}$ for all $e \in T \setminus \{0,1\}$. Clearly $f \neq 0$, so $f = 1$ and $H(1) = \mathrm{BdS}$ is a circle group. Hence $S = \bigcup_{e \in T} H(e)$, where each $H(e)$ is a circle group if $e \neq 0$, and $\mathrm{BdS} = H(1)$. Let $t \in xH(1)$; then $t = xh$ and $tS = xhS = xS$ since $RS = hS = S$, so $t \mathcal{R} x$ and it follows that $xH(1) \subset H(x)$. Now for each $x \in S$, either $xH(1) = x$ or $xH(1) = H(x)$. Suppose $0 \neq f = f^2$ and $H(f) = fH(1)$. Such an f exists since $H(1) = 1 \cdot H(1)$. S is Hausdorff and multiplication is continuous so there exists an open connected set U about f and an open connected set V about x for some $x \in H(f) \setminus \{f\}$, such that $U \cap V = \square$ and $Ux \subset V$. Let $T_1 = \{t \in T : t \leq f\}$ and consider $e \in T_1 \setminus \{f\} \cap U$. Then $ex \in V$, but $x = fh$ for some $h \in H(1)$ and $ef = e$, so $ex = efh = eh$. Now $eh \in eH(1)$ and $eh \neq e$, so $eH(1) = H(e)$.

Let $e = \inf \{t \in T \setminus \{0\} : H(t) = tH(1)\}$ and suppose $e \neq 0$. Then $eH(1) = e \neq 1$ by the argument above. For $x \in H(e)$, $x \neq e$, let U be an open connected set about e and V an open connected set about x such that $U \cap V = \square$. Now there exists a sequence $\{e_n h_n\}$ converging to x such that the sequence $\{e_n\}$ converges to e,

$e_n > e_{n+1}$, $e_n < e$ and $h_n \in H(1)$ for all n. Let $h \in H(1)$ such that a subsequence $\{h_{n_j}\}$ of $\{h_{n_j}\}$ converges to it.

Now $\{e_{n_j}\}$ converges to e , so $\{e_{n_j}h_{n_j}\}$ converges to $eh = e$; but $\{e_{n_j}h_{n_j}\}$ is a subsequence of $\{e_n h_n\}$ which converges to x , a contradiction. Therefore $e = 0$, $S = T\cdot H(1)$ and $N_x = \{h\in H(1): xh = x\}$ is finite for all $x \neq 0$ by [8]. Suppose $fh = f$ for $f \in T\backslash\{0\}$ and $h \in H(1)\backslash\{1\}$; that is, $N_f \neq \{1\}$. For $e < f$ then $ef = e$ and $eh = efh = ef = e$, so f is assumed to be the maximal element in T such that $fh = f$. Let $T_2 = T\backslash T_1 \cup \{f\}$ where $T_1 = \{t \in T: t \leq f\}$. Now $Th = T_1 \cup T_2 h$ is an arc from 0 to h where $T_2 h \cap H(1) = h$. Consider the connected set $T_2 \cup T_2 h$ which separates S and in particular it separates an arc A of $H(1)$ from 0. Let $g \in A\backslash\{1,h\}$ and consider the arc Tg from g to 0. Now either $Tg \cap T_2 \neq \square$ or $Tg \cap T_2 h \neq \square$.



If $e \in Tg \cap T_2$, then $e = \ell g \neq 0$ for some $\ell \in T$. Therefore $e \in \ell H(1) = H(\ell)$, $e = \ell$ and $e = eg$. Hence $fg = feg = fe = f$. If $0 \neq eg = \ell h \in Tg \cap T_2 h$ where $e \in T$ and $\ell \in T_2$, then $e = \ell hg^{-1}$, and $e = e^2 = \ell hg^{-1} \cdot gh^{-1}\ell = \ell^2 = \ell$, so $e = ehg^{-1}$. Now $e = \ell \in T_2$, so $fe = e$. So $fg^{-1} = fhg^{-1} = fehg^{-1} = fe = f$. Therefore $A \subset N_f$, a contradiction, since N_f is finite. Hence $N_f = \{1\}$ for all $f \in T \backslash \{0\}$ and the maps from $H(1)$ onto $fH(1) = H(f)$ defined by mapping h onto fh are homeomorphisms.

Now $fh^{-1} \in fH(1) - H(f)$, $(fh^{-1})^{-1} = hf \in fH(1)$, so $hf \cdot fh^{-1} = hfh^{-1} = H(f)$, but $hfh^{-1} \in T = E$, so $hfh^{-1} = f$ or $hf = fh$. That is to say T is contained in the center of S.

Consider the multiplication map m from $T \times H(1)$ onto $T \cdot H(1) = S$. Now $m(e,g)m(f,h) = egfh = efgh = m(ef,gh)$ so m is a homomorphism. Note that $m^{-1}(0) = 0 \times H(1)$.

Clearly, $\dfrac{T \times H(1)}{0 \times H(1)}$ is homomorphic to S and if $fh = eg$, then as above $f = e$, so $fh = fg$ or $fhg^{-1} = f$. Therefore $hg^{-1} \in N_f$ or $hg^{-1} = 1$ and $h = g$. Hence the canonical map is one to one, thus a homeomorphism and it follows that S is a min cone over a circle.

We now consider the possibility of E having a cut point and $\dim E = 2$. It will be shown that this case does not exist.

Throughout the remainder of this chapter e will denote $\sup\{t \in T : h(t)$ is a circle group$\}$ where T still denotes a min thread from 1 to 0. Since E is assumed to have a cut point, e does exist.

Lemma 2.8. $H(e)$ is a circle group.

Proof: Let $A = \cup H(t)$ for all $t \in T$ such that $t < e$. Each $H(t)$ is a circle group, so A is open since $H(t) \subset \operatorname{int} H(g)$ for $t < g < e$. Clearly A is not dense in the plane, so by Theorem 2.4, $\dim \operatorname{Bd}A = 1$. For $x \in \operatorname{Bd}A$, there exists a sequence $\{x_n\}$ of A converging x such that $x_n x_n^{-1} = x_n^{-1} x_n = e_n$, $e_n < e_{n+1}$ for all n, and the sequence $\{e_n\}$ converges to e. By the continuity of inversion, the sequence $\{x_n^{-1}\}$ converges to x^{-1}. Now both $\{x_n x_n^{-1}\}$ and $\{x_n^{-1} x_n\}$ converge to e, so $xx^{-1} = e = x^{-1}x$, $x = H(e)$ and $\operatorname{Bd}A \subset H(e)$. Hence $\dim H(e) = 1$ and $H(e)$ is a circle group [8].

Remark: Since e is a cut point of E and e separates $E \cap S \setminus SeS = E \cap (\operatorname{ext} H(e))$ from 0, then for any $t \in S \setminus SeS$, $e < tt^{-1}$ and $e < t^{-1}t$.

Lemma 2.8. The idempotents of SeS are contained in the center of S.

Proof: Note that $E \cap SeS = \{f \in T : f \leq e\}$ is a min thread from e to 0 by corollary 2.6. For $f \in E \cap SeS$ and $t \in S \backslash SeS$, $ftS = ftt^{-1}S = fS$, since $f \leq e < tt^{-1}$, so $ft \mathcal{R} f$. Similarly $Stf = St^{-1}tf = Sf$ and $tf \mathcal{L} f$. $H(f)$ is a circle group, so $H(f) = D(f) = R(f) = L(f)$ and it follows that $(tf)^{-1} = ft^{-1} \in H(f)$. Now $tft^{-1} \in E$ and $tft^{-1} = tf \cdot ft^{-1} \in H(f)$. Therefore $tft^{-1} = f$ or $tf = ft$. For $x \in SeS$, either (i) $f \leq xx^{-1} = x^{-1}x$ or (ii) $x^{-1}x = xx^{-1} < f$. In case (i), $fxS = fxx^{-1}S = fS$ and $xfx^{-1} = f$. Hence $xf = fx$.

In case (ii), $fxS = fxx^{-1}S = xx^{-1}S = x^{-1}xS$, so $fxx^{-1} = xx^{-1}$ and $x^{-1}xf = x^{-1}x$. Multiplying by x on the appropriate side implies $fx = x$ and $xf = x$. Hence $fx = x = xf$. Therefore f is an element of the center of S.

Lemma 2.9. If $f \in E$ such that $e < f$ then $H(f) \cdot e$ is a closed subgroup of $H(e)$.

Proof: Clearly $H(f)e$ is closed and is contained in $H(e)$. By the previous lemma, e commutes with all elements of $H(f)$, so for xe, $ye \in H(f)e$ where $x, y \in H(f)$, $(ye)^{-1} = ey^{-1} = y^{-1}e \in H(f)e$ and

$xe(ye)^{-1} = xe \cdot ey^{-1} = xey^{-1} = xy^{-1}e \in H(f)e$. Therefore $H(f)e$ is a closed subgroup of $H(e)$.

Remark: $H(f)$ is not a circle group and $\dim H(f)=0$. Therefore $\dim H(f)e = 0$ and $H(f)e$ is a finite cyclic subgroup [8]. Also note that the map from $H(f)$ into $H(e)$, mapping x onto xe is a homomorphism.

Lemma 2.10. For $t \in S\backslash SeS$ and $x \in H(e)$, $tx = xt$.

Proof: Note $H(e)$ is abelian, $te \in H(e)$ by [14], and $te = et$ by lemma 2.8. So $xe = x = ex$ and $tx = t \cdot ex = te \cdot x = x \cdot te = x \cdot et = xe \cdot t = xt$.

Lemma 2.11. For $t \in S\backslash SeS$ and $x, y \in H(e)$, then $tx = ty$ implies $x = y$.

Proof: $tx = t \cdot ex = t \cdot ey = ty$ and multiplying by $(te)^{-1}$ implies $x = ex = (te)^{-1}te \cdot x = (te)^{-1} \cdot tex = (te)^{-1} \cdot tey = ey = y$.

Lemma 2.12. If $t \in S\backslash SeS$ and $tx = x$ for some $x \in H(e)$, then $th = h$ for all $h \in H(e)$.

Proof: Multiplying $tx = x$ by x^{-1} implies $txx^{-1} = xx^{-1}$ or $te = e$. Therefore $th = teh = eh = h$ for all $h \in H(e)$.

Lemma 2.13. Let $C = \{t \in S \setminus \operatorname{int} H(e) : te = e\}$. Then C is a closed connected inverse subsemigroup such that $C \cap H(e) = \{e\}$.

Proof: Let $\{t_n\}$ be a sequence contained in C converging to t. $\{t_n e\}$ converges to te, but $t_n e = e$ for all n, so $te = e$ and C is closed. Let $T_1 = \{t \in T : t \geq e\}$. For $t \in C$ and $x = ft \in T_1 t$, $xe = fte = fe = e$ and $T_1 t \subset C$. Therefore C is connected. For $t \in C$, $te = e$ and $(te)^{-1} = et^{-1} = e$, but e is in center of S, so $t^{-1}e = e$ and $t^{-1} \in C$. For $x, t \in C$ then $txe = te = e$ and hence $C^2 \subset C$. Therefore C is a compact connected inverse semigroup such that $C \cap H(e) = \{e\}$.

Remark: If f lies on a min thread from 1 to 0 then $M(f) = \{fg \in E : gf = f\}$ is connected.

Lemma 2.14. $1 \notin \operatorname{Bd} C \cap \overline{S \setminus C}$.

Proof: Suppose there is a sequence $\{t_n\}$ converging to 1 such that $t_n \notin C$ for all n. $\dim E = 2$ implies there exists $f \in \operatorname{int} E$ [10, pp.44], and since $E \cap SeS$ is an arc, $f \in S \setminus SeS$ and $fe = e$.

Now the sequence $\{t_n f\}$ converges to f, so there exists an integer m such that $t_m f \in E$ and $t_m f \in S \setminus SeS$. Therefore $t_m \cdot e = t_m \cdot f_e = t_m f \cdot e = e$, a contradiction since

$t_m \not\in C$. Therefore $1 \not\in \operatorname{Bd} C \cap \overline{S\backslash C}$.

Remark: Clearly $C \cdot S\backslash C \subset S\backslash C$ and in particular $aT \cap C = \square$ for all $a \in S\backslash C$. Also note that $\dim C = 2$ since $E \subset C \cup T$ and $\dim T = 1$.

Lemma 2.15. If $aT \cap bT \neq \square$ then there exists $f \in T$ such that (i) $at = bt$ for all $t \leq f$ and (ii) if $T_1 = \{t \in T: t \geq f\}$ then $aT_1 \cap bT_1 = \{af\}$.

Proof: Let $f = \sup\{t \in T: at = bt\}$ and clearly $af = bf$. For $t \leq f$, multiplying $af = bf$ on the right by t yields $at = bt$.

Suppose $x \in aT_1 \cap bT_1$. Then there exists $t \in T_1$, $t_1 \in T_1$ such that $x = at = bt_1$. t and t_1 compare, so without loss of generality suppose $t \leq t_1$. Then $at = at \cdot t = bt_1 t = bt$ and $t \leq f$. But $t \in T_1$, so $t = f$ and $x = af$.

Remark: Note $aT \cap bT$ is connected.

Lemma 2.16. C is acyclic.

Proof: Let $T_1 = \{t \in T: t \geq e\}$ and consider the map H from $T_1 \times C$ onto C defined by $H(t,c) = tc$. Clearly H is continuous and $H(1,c) = c$ and $H(e,c) = e$ for all $c \in C$. The Generalized Homotopy Theorem [18] implies

$H^n(C) \approx H^n(e)$ for all n, hence C is acyclic.

Remark: $BdS \not\subset C$ since $\operatorname{int} H(e) \cap C = \Box$ and C does not cut R^2.

Lemma 2.18. $BdS \cap C$ is connected.

Proof: Let $a \in BdS \cap C$ such that $a \neq 1$. It will suffice to show that one of the arcs of BdS, from a to 1 is contained in C. Let A and B be the arcs from this separation. Suppose neither arc, A or B, is contained in C and consider the set $T \cup aT$. Let $x \in A \setminus C$, $y \in B \setminus C$, $f = \sup\{t \in T: t = at\}$ and $T_1 = \{t \in T: t \geq f\}$. Clearly $f \geq e$. Lemma 2.15 implies $T_1 \cap aT_1 = \{f\}$, so $T_1 \cup aT_1$ is an arc from 1 to a, not contained in BdS. Hence $T_1 \cup aT_1$ separates S and in particular it separates x from y. $H(e) \setminus \{e\} \cap (T_1 \cup aT_1) = \Box$ and $H(e) \setminus \{e\}$ is connected, so either x or y is separated from $H(e) \setminus \{e\}$. $xe \in xT \cap H(e) \setminus \{e\}$ and $ye \in yT \cap H(e) \setminus \{e\}$, a contradiction, since $xT \cup yT \cup H(e) \setminus \{e\}$ is a connected set containing x and y missing C and in particular $T_1 \cup aT_1$. Hence $BdS \cap C$ is connected.

Throughout the remainder of this chapter let A denote the arc $BdS \cap C$ from a to b and let B denote the complement arc $\overline{BdS \setminus C}$.

Lemma 2.19. $b = a^{-1}$ and $aT_1 \cup a^{-1}T_1 = \overline{S\setminus C} \cap C$ where $T_1 = \{t \in T : t \geq e\}$.

Proof: There exists a unique $x \in H(e)\setminus\{e\}$ such that $x^2 = e$ or $x = x^{-1}$. For $s \in S\setminus SeS$, if $se = x$ then $es^{-1} = s^{-1}e = x$, so by lemma 2.11, $s = s^{-1}$.

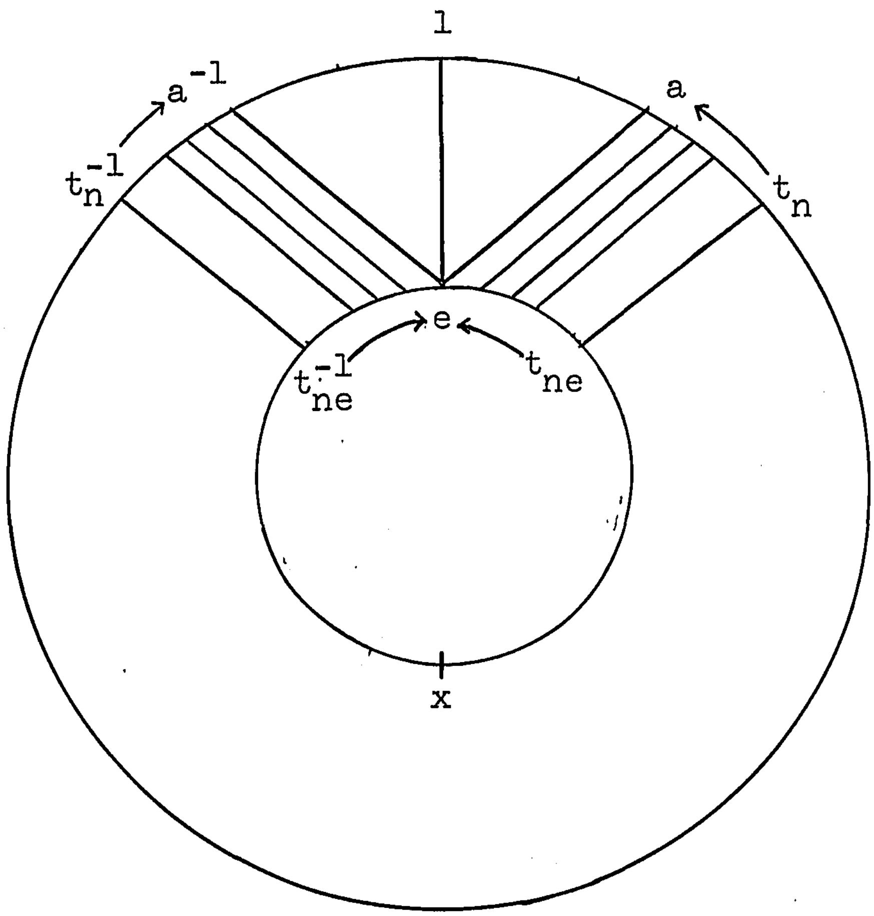
Now if $\{t_n\}$ is a sequence converging to a, $t_n \in S\setminus C$ for all n, then $\{t_ne\}$ converges to e, so we choose a sequence $\{t_n\} \subset B$ converging to a such that $t_n \neq t_n^{-1}$ for all n, $t_ne \neq t_me$ for $n \neq m$, et_n and et_n^{-1} are separated by e and x in $H(e)$, $\{t_n\} = t_nT \cap BdS$, $\{t_n^{-1}\} = t_n^{-1}T \cap BdS$, and $\{t_ne\}$ is contained in one of the two arcs from e to x. Inversion is a homeomorphism, so $\{t_n^{-1}\}$ converges to a^{-1} and both the sequence $\{t_n^{-1}\}$ and a^{-1} are contained in BdS. C is an inverse semigroup by lemma 2.13, so either $a^{-1} = a$ or $a^{-1} = b$ since $\{a,b\} = A \cap B$.

Let H' be the arc of $H(e)$ from e to t_1e which does not contain x and consider the arc $aT_1 \cup t_1T_1 \cup H'$. Note it is an arc since $aT_1 \cap t_1T_1 = \square$, $aT_1 \cap H' = \{e\}$ and $t_1T_1 \cap H' = \{t_1e\} \neq \{e\}$. Now the arc $aT_1 \cup t_1T_1 \cup H'$ separates t_n, for $n > 1$, from $H(e)\setminus H'$ and in general separates any point of B between a and t_1 from $H(e)\setminus H'$.

Consider $t_n^{-1}T_1$ for $n > 1$ and by the choice of the sequence $\{t_n\}$, $t_n^{-1}T_1 \cap t_nT_1 = \Box$, $t_n^{-1}T_1 \cap aT_1 = \Box$, and $t_n^{-1}e \not\in H'$. Now $t_n^{-1}T_1$ is a connected set containing t_n^{-1} and meeting $H(e)\backslash H'$, so t_n^{-1} does not lie between a and t_1 in B.

Therefore the sequence $\{t_n^{-1}\}$ converges to b and $a^{-1} = b$.

<u>Diagram</u>



For $t \in T_1$, the sequence $\{t_n t\}$ converges to at and $t_n t \in S\backslash C$ for all n, so $aT_1 \subset \overline{S\backslash C} \cap C$ and similarly $a^{-1}T_1 \subset \overline{S\backslash C} \cap C$.

Let H_n be the arc of $H(e)$ containing 1 from $t_n e$ to $t_n^{-1}e$ for each n and consider the arcs $C_n = t_n T \cup t_n^{-1} T \cup H_n$. Let B_n be the arc of BdS from t_n to t_n^{-1} contained in B and $A_n = \overline{\text{BdS}\backslash B_n}$. Note $A \subset A_n$ and $A_n \cap B_n = \{t_n, t_n^{-1}\}$. Now clearly $C_n \cup A_n \cup B_n$ is a θ-curve for each n. Denote the bounded regions of the separation of R^2 by each θ-curve by P_n and Q_n where P_n has edges C_n and B_n and Q_n has edges C_n and A_n. From the definitions of the θ=curves it is clear that $C \subset \overline{Q_n}$ for each n and $\overline{Q}_{n+1} \subset \overline{Q}_n$ for all n. Let $Q = \bigcap_{n=1}^{\infty} \overline{Q}_n$ and clearly $Q \cap H(e) = \{e\}$. By lemma 2.15, for $s \in Q$, $sT_1 \cap H(e) = \{e\}$ and $Q = C$ and it follows that $A \cup aT_1 \cup a^{-1}T_1 = \text{BdC}$ and hence $aT_1 \cup a^{-1}T_1 = \overline{S\backslash C} \cap C$.

<u>Theorem 2.20</u>. There does not exist an inverse semigroup on the 2-cell with an identity whose set of idempotents E has a cut point and $\dim E = 2$.

<u>Proof</u>: If such a semigroup S exists, then since $\{t_n^{-1}\}$ converges to a^{-1}, $\{at_n^{-1}\}$ converges to aa^{-1}.

Now $\{at_n^{-1}\} \subset S\backslash C$, so $aa^{-1} \in \overline{S\backslash C} \cap C = aT_1 \cup a^{-1}T_1$.
Therefore either $aa^{-1} \leq a$ and $a^{-1} \leq a^{-1}a$ or $aa^{-1} \leq a^{-1}$ and $a \leq a^{-1}a$. Hence in either case $a = a^{-1} \in E$ and $C = T_1$, a contradiction since $\dim C = \dim E = 2$.

Combining Theorem 2.7 and Theorem 2.20, we have established the following main result.

Theorem 2.21. If S is an inverse semigroup with an identity on the 2-cell whose set of idempotents E has a cut point then S is a min cone over a circle and E is a min thread.

Concluding Remarks: The following result is a consequence of Theorems 2.20, 2.7 and corollary 2.6.

Theorem 2.22. If S is an inverse semigroup with an identity on an annulus whose kernel K is one of the boundary circles, then S is iseomorphic to $K \times E$ where the set of idempotents E is a min thread.

CHAPTER III

Introduction. In this chapter we investigate inverse semigroups on the 2-cell with an identity 1 and whose set of idempotents E has no cut point. The inversion homeomorphism is shown to be the identity map, and it then follows that S must be a commutative Clifford semigroup. The maximal group $H(1)$ containing 1 then must be a subgroup, not necessarily proper, of the four group, $Z_2 \times Z_2$. Examples are given for all possible cases of $H(1)$.

In the special case when $M(e) = \{f \in E : e \leq f\}$ is connected for each $e \in E$ and $H(1) \approx Z_2 \times Z_2$, then the semigroup S must be the continuous monotone homomorphic image of the (min cone over Z_2)X(min cone over Z_2).

Preliminaries and Results. Throughout this chapter, S will denote an inverse semigroup on the 2-cell with an identity 1 and whose set of idempotents E has no cut point.

By [13], E is a retract and by Theorem 1.7, E is a 2-cell. T will denote a min thread from 1 to 0 and let $C = \{s \in S : s = s^{-1}\}$. It is clear that $T \subset E \subset C$.

Lemma 3.1. C is closed.

Proof: Let $x \in S$ and $\{x_n\}$ a sequence of elements of C converging to x. Now $x_n = x_n^{-1}$, so $\{x_n^{-1}\}$ is a sequence converging to x also, but since inversion is continuous, $\{x_n^{-1}\}$ converges to x^{-1}. Hence $x = x^{-1}$ and $x \in C$. Now it follows that C is closed.

Lemma 3.2. For $c \in C \cap \mathrm{Bd}S$, let U be an open connected disc about c such that $\overline{U}$ is a 2-cell, $\overline{U} = \overline{U}^{-1}$, and $\overline{U} \cap \mathrm{Bd}S$ is an arc from a to b containing c. Denote $\overline{U} \cap \mathrm{Bd}S$ by A and let A_1 be the arc of A from a to c and A_2 the arc of A from c to b. If $\{t_n\}$ is a sequence in $A_1 \backslash \{a, c\}$ converging to c such that $t_n \notin C$ for all n, then there exists an integer $N \geq 1$ such that $t_n^{-1} \in A_2 \backslash \{b, c\}$ for all $n \geq N$.

Proof: Suppose there is a subsequence $\{t_{n_j}\} \subset \{t_n\}$ such that $\{t_{n_j}^{-1}\} \subset A_1 \backslash \{a, c\}$. Clearly both $\{t_{n_j}\}$ and $\{t_{n_j}^{-1}\}$ converge to c. Consider the subarcs of $A_1 \backslash \{a, c\}$ from t_{n_j} to $t_{n_j}^{-1}$, denoted by $A(j)$.

The inversion map i is a homeomorphism, mapping $\mathrm{Bd}S$ onto $\mathrm{Bd}S$, c onto c and, in particular, A onto A. Therefore, the connected set $i(A(j))$ is contained in $A \backslash \{1\}$ since $i(1) = i^{-1}(1) = 1 \notin A(j)$ for all j, and $\{t_{n_j}, t_{n_j}^{-1}\} \subset A(j) \subset i(A(j)) \subset A_1 \backslash \{a, c\}$. Now homeomorphisms preserve cut points, so $i(A(j)) = A(j)$ for all j.

The restriction of i to $A(j)$ has a fixed point, so there exists $x_j \in A(j)\setminus\{t_{n_j}, t_{n_j}^{-1}\}$ such that $i(x_j) = x_j$ or $x_j = x_j^{-1}$. Clearly x_j and c separate t_{n_j} from $t_{n_j}^{-1}$ in BdS. The convergence of $\{t_{n_j}\}$ to c and $\{t_{n_j}^{-1}\}$ to c implies the existence of x_h and x_k, elements of $A_1\setminus\{a,c\}$ such that $x_h = x_h^{-1}$, $x_k = x_k^{-1}$, and x_h and x_k separate t_{n_h} from $t_{n_h}^{-1}$ in BdS. That is, the arc from x_h to x_k contained in $A_1\setminus\{a,c\}$ contains either t_{n_h}, or $t_{n_h}^{-1}$ but not both.

Now the homeomorphism i maps this arc from x_h to x_k onto itself, a contradiction since the arc contains one but not both of t_{n_h} and $t_{n_h}^{-1}$. Therefore, $\{t_n^{-1}\}$ is residually in $A_2\setminus\{c,b\}$ and the proof is complete.

Corollary 3.3. Either $\text{BdS} \subset C$ or $\text{BdS} \cap C$ contains no arcs.

Proof: Suppose $\text{BdS} \not\subset C$ and suppose there exists $c \in \text{BdS} \cap C$ such that c is an end point of an arc in $\text{BdS} \cap C$ and $c \in \overline{\text{BdS}\setminus C}$.

Now let U be an open connected disc about c as described in the lemma, such that $A_2 \subset C$ and $A_1 \not\subset C$. Let $\{t_n\}$ be a sequence converging to c such that $\{t_n\} \subset A_1 \backslash C$. Therefore, $t_n \neq t_n^{-1}$ for all n and by the lemma $\{t_n^{-1}\}$ is residually in A_2, a contradiction, for if $t_n^{-1} \in A_2$ then $t_n^{-1} = t_n$. Therefore, $\mathrm{Bd}S \cap C$ contains no arcs, or, it is totally disconnected if $\mathrm{Bd}S \not\subset C$. We proceed now to show that the inversion homeomorphism is the identity map. That is $C = S$. The first step is to establish that $\mathrm{Bd}S \subset C$. This will be done by considering the following two cases:

(1) if $T \backslash \{1\} \cap \mathrm{Bd}S \neq \Box$ and

(2) if $T \backslash \{1\} \cap \mathrm{Bd}S = \Box$.

Case 1: For $t \in T \backslash \{1\} \cap \mathrm{Bd}S$, let $T_1 = \{t_1 \in T : t_1 \geq t\}$ and let A and B be the two arcs of $\mathrm{Bd}S$ from 1 to t such that $A \cup B = \mathrm{Bd}S$ and $A \cap B = \{1, t\}$. Note that if $\mathrm{Bd}S \not\subset C$, then $A \neq T_1$ and $B \neq T_1$. That is, $T_1 \cap \mathrm{int}\, S \neq \Box$.

Lemma 3.4. Either $aT \cap T_1 \neq \Box$ for all $a \in A$ or $bT \cap T_1 \neq \Box$ for all $b \in B$.

Proof: Suppose there exists $a \in A \backslash \{1, t\}$ and $b \in B \backslash \{1, t\}$ such that $aT \cap T_1 = \Box = bT \cap T_1$, or, that $(aT \cup bT) \cap T_1 = \Box$. Since T_1, A, and B are all arcs from 1 to t such that $a \notin T_1$ and $b \notin T_1$, it follows $T_1 \not\subset BdS = A \cup B$ or that $T_1 \cap \text{int } S \neq \square$. Therefore, by Theorem 1.1, T_1 cuts S and separates a from b by Theorem 1.2. Now $0 \in aT \cap bT$, so $aT \cup bT$ is a connected set containing both a and b. But $(aT \cup bT) \cap T_1 = \square$ is a contradiction.

Remark: Without loss of generality, we may assume $aT \cap T_1 \neq \square$ for all $a \in A$. Note also that the arcs aT and bT in the corollary may be replaced with the arcs Ta and Tb.

Corollary 3.5. If $af \in aT \cap T_1$ for $f \in T$, then $f \geq t$.

Proof: If $f < t$, then $af \cdot t = a \cdot ft$, and since $af \in T_1$, $af \geq t$ and $af \cdot t = t$. So $af = t$, and since $af \in E$, $af \leq f$ or $t \leq f$, a contradiction. Therefore, $f \geq t$ and $aT \cap T_1 = aT_1 \cap T_1$.

Lemma 3.6. For case 1, $BdS \subset C$.

Proof: Suppose $BdS \not\subset C$ and by corollary 3.3 $BdS \cap C$ contains no arcs. If follows that T_1 separates S. Let $\{t_n\}$ be a sequence converging to 1 contained in A, $t_n \notin C$ for all n, and such that the sequence $\{t_n^{-1}\}$

is contained in B. Such a sequence exists since $\mathrm{Bd}S \cap C$ is totally disconnected and by lemma 3.2 the inverses are in B. Clearly $\{t_n^{-1}\}$ converges to 1. Let $f_n = \sup\{t_1 \in T_1 : t_n t_1 \in T_1\}$. The existence of f_n is clear from the assumption that $aT_1 \cap T_1 \neq \square$ for all $a \in A$.

Now it follows that $t_n f_n \in T_1$, and since $t_n f_n \in E$, $t_n f_n = (t_n f_n)^{-1} = f_n t_n^{-1}$ and $T_1 t_n^{-1} \cap T_1 = \square$. Clearly $f_n = \sup\{t_1 \in T_1 : t_1 t_n^{-1} \in T_1\}$ and $f_n \neq 1$ for all n. Let $T_n = \{t_1 \in T_1 : t_1 \geq f_n\}$ and by the definition of f_n , $t_n T_n \cap T_1 = \{t_n f_n = f_n t_n^{-1}\} = T_n t_n^{-1} \cap T_1$. Let $C(t_n^{-1})$ be the components of S T_1 containing t_n and t_n^{-1} respectively. By lemma 1.2, $C(tn) \cap C(t_n^{-1}) = \square$ for all n.

Since $T_n \backslash f_n$ is a connected set and $t_n \cdot (T_n \backslash f_n) \cap T_1 = \square$, $t_n (T_n \backslash f_n) \subset C(t_n)$ for all n and similarly $(T_n \backslash f_n) t_n^{-1} \subset C(t_n^{-1})$ for all n. For $h \in T_n$, if $t_n h \in t_n T_n \cap E$, then $t_n h = (t_n h)^{-1} = h t_n^{-1}$ and $t_n h \in t_n T_n \cap T_n t_n^{-1}$. But $t_n T_n \cap T_n t_n^{-1} = \{t_n f_n\}$, so $t_n T_n \cap E = \{t_n f_n = f_n t_n^{-1}\} = T_n t_n^{-1} \cap E$. Now $K_n = t_n T_n \cup T_n t_n^{-1}$ is an arc from t_n to t_n^{-1} such that $K_n \cap E = \{t_n f_n\}$.

We consider now the two cases:

(i) there exists an n such that $f_n t_n \neq t$ or

(ii) $t_n f_n = t$ for all n.

Case (i). Since $1 \notin K_n$ and $t \notin K_n$, $K_n \not\subset \mathrm{Bd}S$ if $t_n f_n \neq t$, so K_n separates S and in particular K_n separates 1 from t. It follows that K_n separates E. Therefore, $\{t_n f_n\} = K_n \cap E$ is a cut point of E, a contradiction.

Case (ii). Suppose $t_n f_n = t$ for all n. If $t_1 \in (T_1 \cap \mathrm{Bd}S)\setminus\{1,t\}$, then either $t_1 \in A\setminus\{1,t\}$ or $t_1 \in B\setminus\{1,t\}$. Letting $t_1 \in B$ without loss of generality, the arc $T(t_1)$ from 1 to t_1 of T_1 separates S and would also separate t_n^{-1} from t for some n. But $T_n t_n^{-1}$ is a connected set from t_n^{-1} to t such that $T_n t_n^{-1} \cap T_1 = \{f_n t_n^{-1} = t\}$, so $T(t_1) \subset T_1 \setminus t$, $T(t_1) \cap T_n t_n^{-1} = \square$ and t_n^{-1} and t lie in the same component of $S \setminus T(t_1)$, a contradiction. Therefore, $T_1 \cap \mathrm{Bd}S = \{1,t\}$ and T_1 separates S into exactly two components, C_A containing the arc $A\setminus\{1,t\}$ and C_B containing the arc $B\setminus\{1,t\}$.

If $1 = \sup\{f_n \in T_1\}$, then since $f_n \neq 1$ for all n, there is a subsequence $\{f_{n_j}\}$ converging to 1. Now $\{t_{n_j}\}$ converges to 1, so $\{t_{n_j} f_{n_j}\}$ converges to 1, a contradiction since $t_{n_j} f_{n_j} = t$ for all j and $t \neq 1$. Therefore, $e < 1$ where $e = \sup\{f_n \in T_1\}$. Let f be any element of $T_1 \setminus \{1\}$ larger than e and by the definition of the f_n's, $t_n f$ and $f t_n^{-1}$ are not idempotents. Note $t_n f \in t_n T_1 \setminus t \subset C_A$ and $f t_n^{-1} \in T_1 t_n^{-1} \setminus t \subset C_B$.

If $f \in \operatorname{int} E$, then there is an open set U about f such that $U \subset E$. Since $\{t_n f\}$ converges to f, $\{t_n f\}$ is residually in U, a contradiction, since $t_n f \notin E$ for all n.

Therefore, the arc of T_1 from 1 to e is contained in BdE, a simple closed curve, and in particular $f \in$ BdE.

There exists an open set U about f such that either $U \cap C_A \subset E$ or $U \cap C_B \subset E$. The sequence $\{t_n f\}$ is residually in $U \cap C_A$ and the sequence $\{f t_n^{-1}\}$ is residually in $U \cap C_B$, a contradiction since $\{t_n f\} \cap E = \square = \{f t_n^{-1}\} \cap E$. Therefore $\mathrm{Bd}S \subset C$ under the conditions of case 1.

Case 2. Suppose $T\setminus\{1\}\cap \mathrm{Bd}S = \square$ and let U be an open connected disc about 1, such that $\overline{U}$ is a 2-cell, $\overline{U}^{-1} = \overline{U}$, $0 \notin \overline{U}$, and $\overline{U} \cap \mathrm{Bd}S$ is an arc A from a to b where $1 \in A\setminus\{a,b\}$. Let A_1 be the arc of A from a to 1 and let A_2 be the arc of A from 1 to b.

Lemma 3.7. In case 2, $\mathrm{Bd}S \subset C$.

Proof: Suppose $\mathrm{Bd}S \not\subset C$ and then by corollary 3.3 $\mathrm{Bd}S \cap C$ contains no arcs. Hence there is a sequence $\{t_n\} \subset A_1\setminus\{a,1\}$, $t_n \notin C$ for all n, converging to 1 and by lemma 3.2, the sequence $\{t_n^{-1}\}$ is residually in $A_2\setminus\{1,b\}$, also converging to 1. Let $t = \sup\{t_1 \in T: t_1 \in \overline{U}\setminus U\}$ and $T_1 = \{t_1 \in T: t_1 \geq t\}$. Clearly $t \neq 1$ and T_1 is a subarc of T spanning $\overline{U}$ from 1 to t, separating $\overline{U}$ into exactly two components R_1 containing $A_1\setminus\{a,1\}$ and R_2 containing $A_2\setminus\{1,b\}$. Let $f_n = \sup\{t_1 \in T: t_n t_1 \in \overline{U}\setminus U\}$. Then $t_n f_n \in \overline{U}\setminus U$, and it follows that $(t_n f_n)^{-1} = f_n t_n^{-1} \in \overline{U}\setminus U$ and $f_n = \sup\{t_1 \in T: t_1 t_n^{-1} \in \overline{U}\setminus U\}$ since inversion is a homeomorphism.

For all n, $f_n \neq 1$ since $t_n \notin \overline{U}\setminus U$. Let $T_n = \{t_1 \in T: t_1 \geq f_n\}$ and clearly $t_n \cdot T_n\setminus\{f_n\}$ and $T_n\setminus f_n \cdot t_n^{-1}$ are contained in U. If there exists an n

such that $t_n T_n \cap E = \Box$, let $f = \sup\{t_1 \in T_n : t_n t_1 \in E\}$. Then $t_n f = (t_n f)^{-1} = f t_n^{-1}$, $f \neq 1$, and if $T_n' = \{t_1 \in T_n : t_1 \leq f\}$, then $t_n(T_n' \setminus f) \cap E = \Box = (T_n' \setminus f) t_n^{-1} \cap E$, $t_n(T_n' \setminus f) \subset R_1$, and $(T_n' \setminus f) t_n^{-1} \subset R_2$. Therefore $t_n f = f t_n^{-1} \in T_1$. If $t_n f = 1$, then $t_n^{-1} \cdot t_n f = t_n^{-1} \in E$, and hence $t_n^{-1} = t_n$, a contradiction. Therefore $t_n f \neq 1$. The arc $K_n = t_n T_n' \cup T_n' t_n^{-1}$ from t_n to t_n^{-1} is contained in $\overline{U}$ such that $t_n T_n' \cap T_n' t_n^{-1} = \{t_n f = f t_n^{-1}\}$ and $t_n T_n' \cap E = \{t_n f\} = T_n' t_n^{-1} \cap E$. Clearly $K_n \not\subset A$ since $1 \notin K_n$, so K_n separates $\overline{U}$ and S and in particular it separates 1 from the complement of $\overline{U}$ which contains 0. Therefore, it separates E and $t_n f$ is a cut point of E , a contradiction. Therefore for all n , $t_n T_n \cap E = \Box$. Let $e = \sup\{f_n \in T\}$. If $e = 1$, then since $f_n \neq 1$ for all n , there is a sequence $\{f_{n_j}\}$ converging to 1. But the sequence $\{t_{n_j} f_{n_j}\}$ converges to 1 and also it is contained in $\overline{U} \setminus U$, a contradiction since U was chosen open about 1. Therefore $e \neq 1$.

For $f \in T_1$, between 1 and e, $t_n f \in R_1$, $f t_n^{-1} \in R_2$, and $\{t_n f, f t_n^{-1}\} \cap E = \Box$ for all n.

Consider the two cases when either the arc of T_1 from 1 to e is contained in BdE , or there is an f between 1 and e contained in the interior of E.

By the same arguements as presented at the end of case 1 in lemma 3.6, a contradiction is reached in both cases. Therefore, $BdS \subset C$ for case 2. By combining the two cases, the following theorem has been established.

<u>Theorem 3.8</u>. $BdS \subset C$.

We proceed now to one of the main results of this chapter.

<u>Theorem 3.9</u>. $S = C$ or the inverse homeomorphism is the identity map.

<u>Proof</u>: Define a map m from $C \times T$ onto C by the following: $m(c,t) = tct$. Clearly m is continuous and $(tct)^{-1} = t^{-1}c^{-1}t^{-1} = tct$ for all $c \in C$. Note that m is an action since

$$m(c,th) = thcth = thcht = m(hch,t) = m(m(h,c),t).$$

For all $c \in C$, $m(c,1) = c$ and $m(c,0) = 0$. Hence by the generalized homotopy theorem, C is acyclic [18]. Therefore, C does not cut R^2 , and since $BdS \subset C$, $S = C$.

Corollary 3.10. S is commutative.

Proof: For any two elements a and b of S, $a = a^{-1}$, $b = b^{-1}$, and $ab = (ab)^{-1}$ so

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba.$$

Corollary 3.11. S is a Clifford semigroup. That is, S is the union of groups.

Proof: For all $a \in S$, $aa^{-1} = a^2 = a^{-1}a$, so the right and left units of each element of S are equal. The corollary now follows from [5, pp.41]. We now turn our attention to the maximal group $H(1)$. By the previous theorem and corollaries, $H(1)$ is commutative and every element has order 2.

Lemma 3.12. The order of $H(1)$ is less than or equal to 4.

Proof: Using the notation that $O(G)$ denotes the order of a group G, suppose $O(H(1)) > 4$. Hence there are elements a, b, and c of $H(1)$ different from 1, such that $ab \neq c$. Let G be the subgroup of $H(1)$ generated by these three elements. It follows that $O(G) = 8$ and $G \subset \text{BdS}$. Let x be one of the two elements of G such that x and 1 separate BdS into an arc A having only one element y of G between 1 and x, and the complementary arc B containing 5 elements of G between 1 and x. Multiplication by x is a homeomorphism, implying either $xA = A$ or $xA = B$. Remembering that $xG = G$, $xA \neq B$, since xA contains 3 elements of G and B contains 7 elements of G. But if $xA = A$, then $xy = y$, a contradiction. Therefore $O(H(1)) \leq 4$.

Corollary 3.13. Either (i) $H(1) = \{1\}$

(ii) $H(1) \approx Z_2$

or (iii) $H(1) \approx Z_2 \times Z_2$.

Proof: The corollary follows immediately from the theory of groups, the preceding lemma, and the fact that all elements are of order 2.

We now give examples showing the existence of an inverse semigroup of the type investigated in this chapter in each of the three cases for $H(1)$.

First we define two inverse semigroups on an interval [13]. Let I_1 be the inverse semigroup defined on the closed interval $[-1,1]$ by

$$ab = \begin{cases} \min\{|a|,|b|\} & \text{if } ab \geq 0 \\ -\min\{|a|,|b|\} & \text{if } ab < 0. \end{cases}$$

I_1 is a min cone over Z_2 or $I_1 \approx \dfrac{I \times \{-1,1\}}{0 \times \{-1,1\}}$ where I is a min thread and $\{-1,1\}$ is the 2 element group.

Let I_2 be the inverse semigroup defined on the closed interval $[-1,1]$ by

$$ab = \begin{cases} \min\{a,b\} & \text{if } ab \geq 0 \\ \min\{|a|,|b|\} & \text{if } ab < 0. \end{cases}$$

I_2 is a min cone over a 2 element semilattice or

$I_2 \approx \frac{I \times \{0,1\}}{0 \times \{0,1\}}$ where $\{0,1\}$ has the usual semilattice multiplication. Note that I_2 is a semilattice.

Example 1. $S_1 = I_1 \times I_1$

Properties: (1) $H(1) \approx Z_2 \times z_2$ (2) $0 \in \operatorname{int} S$, (3) $E = \{(a,b): a \geq 0 \text{ and } b \geq 0\}$, (4) $H(a,b) \approx Z_2 \times Z_2$ for all (a,b) such that $a \neq 0$, $b \neq 0$, and (5) $S_1 = \cup (i,j)E$ where $(i,j) \in H(1)$ and i and j take on values of 1 and -1.

Example 2. $S_2 = I_1 \times I$ or equivalently the inverse subsemigroup of S_1, $S_2' = \{(a,b) \in S_1 : b \geq 0\}$.

Properties: (1) $H(1) \approx Z_2$, (2) $E = \{(a,b): a \geq 0\}$, (3) $S_2 = E \cup (-1,1)E$, (4) $E \cap (-1,1)E \approx I$, (5) $0 \in \operatorname{Bd} S_2$ and (6) $H(a,b) \approx Z_2$ for all elements (a,b) such that $a \neq 0$.

Example 3. $S_3 = I_1 \times I_2$

Properties: (1) $H(1) \approx Z_2$, (2) $0 \in \operatorname{int} S_3$, (3)

$E = \{(a,b): a \geq 0\}$ (4) $S_3 = E \cup (-1,1)E$, (5) $E \cap (-1,1)E \approx I_2$ and (6) $H(a,b) \approx Z_2$ for all elements (a,b) such that $a \neq 0$.

Example 4. $S_4 = \{(a,b) \in S_1 : b \geq -1/2\}$.

Properties: (1) $H(1) \approx Z_2$, (2) $0 \in \operatorname{int} S_4$, (3) $E = \{(a,b): a \geq 0$ and $b \geq 0\}$ and (4) S_4 contains groups of orders 4 and 2.

Example 5: $S_5 = \{(a,b) \in S_1 : -3/2 \leq b-a \leq 3/2\}$.

Properties: (1) $H(1) \approx Z_2$, (2) $0 \in \operatorname{int} S_5$, (3) $E = \{(a,b): a \geq 0$ and $b \geq 0\}$ and, (4) S_5 contains groups of order 4 and 2.

Example 6: $S_6 = \{(a,b) \in S_1 : a \geq -1/2$ and $b \geq 0\}$.

Properties: (1) $H(1) = \{1\}$, (2) $E = \{(a,b): a \geq 0\}$, (3) $S_6 = E \cup (-1/2,1)E$, (4) $E \cap (-1/2,1)E \approx I$, (5) $0 \in \operatorname{Bd} S_6$ and (6) $H(a,b) \approx Z_2$ for all elements (a,b) such that $a \neq 0$ and $a \leq 1/2$.

Example 7: $S_7 = \{(a,b) \in S_3 : a \geq -1/2\}$.

Properties: (1) $H(1) = \{1\}$, (2) $0 \in \operatorname{int} S_7$, (3) $E = \{(a,b): a \geq 0\}$, (4) $S_7 = E \cup (-1/2,1)E$, (5) $E \cap (-1/2,1)E \approx I_2$ and (6) $H(a,b) \approx Z_2$ for all (a,b) such that $a \neq 0$ and $a \leq 1/2$.

We now prove two lemmas which will lead to the establishment of the fact that $0 \in \overline{S\setminus E}$ if $H(1) \approx Z_2$ or $H(1) \approx Z_2 \times Z_2$.

<u>Lemma 3.14</u>. If $a \in H(1)\setminus\{1\}$, then $aT \cap E \subset T$, and if $at \in aT \cap E$ for $t \in T$, then $at = t$.

<u>Proof</u>: Let $f \in aT \cap E$. Then $f = at$ for some $t \in T$ and $at \in E$. So, $f = at = (at)^2 = a^2t^2 = t$ and $f = at = t \in T$.

<u>Lemma 3.15</u>. For $f \in T\setminus\{0\}$, let $C_f = \{S \in S: sf = f\}$. Then C_f is a compact, connected inverse subsemigroup such that $C_f \cap \mathrm{Bd}S$ is connected.

<u>Proof</u>: Clearly C_f is a compact inverse subsemigroup. Let $T_1 = \{t \in T: t \geq f\}$. For $s \in C_f$, sT_1 is connected and contains both s and f. If $st \in sT_1$ the $tf = f$ and $st \cdot f = sf = f$, so $sT_1 \subset C_f$ and hence C_f is connected.

Define the map m from $C_f \times T_1$ onto C_f by $m(c,t) = ct$. Now $m(ckth) = cth = m(m(c,t),h)$, so T_1 acts on C_f. $m(c,1) = c$ and $m(c,f) = f$ for all $c \in C_f$, hence it follows that C_f is acyclic. Since $C_f \neq S$, $\mathrm{Bd}S \not\subset C_f$.

Suppose $BdS \cap C_f$ is not connected, and let p and q be elements of two different components of $BdS \cap C_f$. Let A and B be the two arcs of BdS from p to q. It follows that there exists $a \in A$ and $b \in B$ such that $\{a,b\} \cap C_f = \Box$. Let $g = \sup\{t \in T: at = bt\}$ and let $T_0 = \{t \in T: t \geq g\}$, and hence $aT_0 \cap bT_0 = \{ag = bg\}$ and by a straight forward argument, $(aT_0 \cup bT_0) \cap C_f = \Box$.

Since $\{p,q\} \cap (aT_0 \cup bT_0) = \Box$, $aT_0 \cup bT_0 \not\subset BdS$, and hence $aT_0 \cup bT_0$ is an arc from a to b separating S and in particular separating p from q , a contradiction, since C_f is connected containing both p and q and $(aT_0 \cup bT_0) \cap C_f = \Box$. Therefore, $C_f \cap BdS$ is connected.

<u>Theorem 3.16</u>. If $H(1) \neq \{1\}$ then $0 \in \overline{S \setminus E}$.

<u>Proof</u>: It is necessary to consider only the case when $aT \cap E = aT \cap T \neq \{0\}$, for all $a \in H(1) \setminus \{1\}$. Since $O(H(1)) = 4$ or 2, there exists $a \in H(1) \setminus \{1\}$, and a decomposition of BdS into two arcs A_1 and A_2 from 1 to a. Let $f = \sup\{t \in T: at = t\}$ and it follows that $af = f \neq 0$.

Now multiplication by a is a homeomorphism, so either $aA_1 = A_1$ or $aA_1 = A_2$.

<u>Case 1</u>. Suppose $aA_1 = A_1$. The arc A_1 has the

fixed point property, so there exists $x \in A_1 \setminus \{1,a\}$ such that $ax = x$, and clearly $axt = xt$ for all $xt \in xT$. Similarly there exists $y \in A_2 \setminus \{1,a\}$ such that $ay = y$ and $ayt = yt$ for all $yt \in yT$. It is clear that $(yT \cup xT) \cap \mathrm{Bd} S = \{x,y\}$, since each arc A_1 and A_2 can have only one fixed point under the multiplication map by a. Let the arcs X and Y from x to y be the decomposition of $\mathrm{Bd} S$ such that $1 \in X$ and $a \in Y$. Let $e = \sup\{t \in T: st = yt\}$ and let $T_e = \{t \in T: t \geq e\}$.

Consider the arc $A = xT_e \cup yT_e$ from x to y which certainly separates S into exactly two regions, R_1 with boundary $X \cup A$, and R_2 with boundary $Y \cup A$. Note that $A \cup X \cup Y$ is a θ-curve.

Multiplication by a maps X onto Y, Y onto X, and R_1 onto R_2. Therefore, $A = \{s \in S: as=s\} = xT \cup yt$, $e = 0$ and $xT \cap yT = \{0\}$.

Since $ax^2 = x^2$ and $ay^2 = y^2$, either $x^2 \in xT$ and $x^2 \leq x$ or $y^2 \in yT$ and $y^2 \leq y$, or, $x = x^2$ or $y = y^2$ since S is stable. Without loss of generality, assume x is idempotent. So $xT \subset E$. Note that if either x or y is zero then the other one must be idempotent, so assume $x \neq 0$.

Let $t \in xT \setminus \{x,0\}$ and suppose $t \in \mathrm{int}\, E$. Let U

be an open set about t contained in E such that $aU = U$. Let $e \in U \cap R_1$. Now $ae \in U \cap R_2 \subset E$, so $ae = (ae)^2 = a^2e^2 = e$, a contradiction, since $R_1 \cap R_2 = \Box$. Therefore, $xT \subset \overline{S \setminus E}$ and in particular $0 \in \overline{S \setminus E}$.

Case 2. Suppose $aA_1 = A_2$. Under this multiplication by a, there is not a point x in BdS such that $ax = x$. Since $0 = a \cdot 0$, $0 \in \text{int } S$.

Consider C_f which contains a and 1 but not BdS. Since $C_f \cap BdS$ is connected, lemma 3.15, either $A_1 \subset C_f$ or $A_2 \subset C_f$. But C_f is a subsemigroup, so either $A_1 \cup aA_1 \subset C_f$ or $A_2 \cup aA_2 \subset C_f$, a contradiction, since $aA_1 = A_2$ and $A_1 \cup aA_1 = A_2 \cup aA_2 = BdS$. Therefore, $aT \cap E = aT \cap T = \{0\}$ and the theorem is established.

Remark: If $H(1) \approx Z_2 \times Z_2$, then there exists $c \in H(1) \setminus \{1\}$ such that $C_1 \setminus \{c,1\} \cap H(1) \neq \Box \neq C_2 \setminus \{c,1\} \cap H(1)$ where C_1 and C_2 are the two arcs of BdS from 1 to c. It follows that $cC_1 = C_2$, and from the proof of case 2, $0 \in \text{int } S$ and $cT \cap E = cT \cap T = \{0\}$.

Section I

Throughout this section, $H(1) \approx Z_2 \times Z_2$ and a, b, and c will denote the elements of $H(1) \setminus \{1\}$ such that:

(i) The decomposition of BdS into arcs A_1 and A_2 from 1 to a, has the properties that $aA_1 = A_1$, $aA_2 = A_2$ and $\{c, b\} \subset A_2 \setminus \{1, a\}$,

(ii) The decomposition of BdS into arcs B_1 and B_2 from 1 to b, has the properties that $bB_1 = B_1$, $bB_2 = B_2$ and $\{a, c\} \subset B_2 \setminus \{1, b\}$, and

(iii) The decomposition of BdS into arcs C_1 and C_2 from 1 to c, has the properties that $cC_1 = C_2$, $cC_2 = C_1$, $a \in C_1 \setminus \{1, c\}$ and $b \in C_2 \setminus \{1, c\}$.

Theorem 3.17. If $H(1) \approx Z_2 \times Z_2$ then:

(1) There exists $e \in A_1$ and $f \in B_1$ such $ae = e$, $bf = f$, $ce \in C_2 \cap B_2$, and $cf \in C_1 \cap A_2$.

(2) $eT \cup ceT = \{s \in S : as = s\}$, $(eT \cup ceT) \cap \mathrm{BdS} = \{e, ce\}$, and $eT \cap ceT = \{0\}$.

(3) $fT \cup cfT = \{s \in S : bs = s\}$, $(fT \cup cfT) \cap \mathrm{BdS} = \{f, cf\}$, and $fT \cap cfT = \{0\}$.

(4) $eT \cap fT = \{0\}$

(5) $eT \cap cfT = \{0\}$ and dually $fT \cap ceT = \{0\}$.

(6) If Q denotes the 2-cell bounded by the arc of BdS from e to f containing 1 and $eT \cup fT$, then $S = Q \cup aQ \cup bQ \cup cQ$, $Q \cap aQ = eT$, $Q \cap bQ = ft$,

$aQ \cap cQ = cfT$, $bQ \cap cQ = ceT$, and $Q \cap cQ = \{0\} = aQ \cap bQ$.

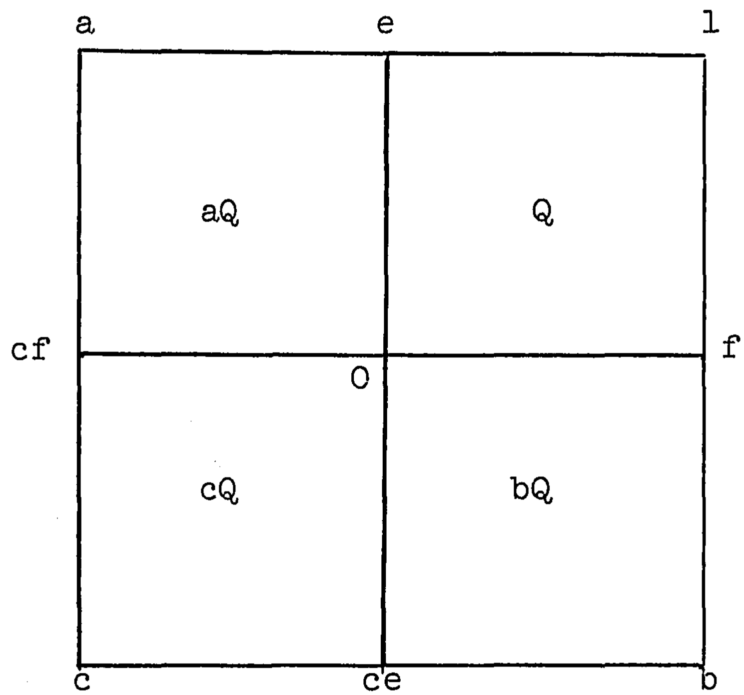
(7) $\{s \in S : cs = s\} = \{0\}$.

(8) $e \in E$ and $f \in E$

(9) $eT \cup fT = BdE \cap \overline{S \setminus E}$ and

(10) $E \subset Q$.

Diagram



Note that $A_1 \subset B_2$, $B_1 \subset A_2$, $A_1 \subset C_1$, $B_1 \subset C_2$, $C_2 \subset A_2$, and $C_1 \subset B_2$.

Proof of 1, 2, and 3: From case 1, Theorem 3.16, there is a unique element e of A_1 such that $ae = e$. Since $c = ab$ and $ac = b$, $ce = abe = be$, $ce \in cA_1 \subset cC_1 = C_2$, and $ce = be \in bA_1 \subset bB_2 = B_2$. Therefore, $ce \in C_2 \cap B_2$ and $a \cdot ce = be = ce$. Hence ce is is the unique element of A_2 such that $ace = ce$. The proof of case 1 of Theorem 3.16 implies $eT \cup ceT = \{s \in S : as = s\}$ and $eT \cap ceT = \{0\}$. Clearly $(eT \cup ceT) \cap BdS = \{e, ce\}$.

Similarly there exists a unique element f of B_1 such that $bf = f$, $bcf = cf$, $cf \in C_1 \cap A_2$, $fT \cup cfT = \{s \in S : bs = s\}$, $(fT \cap cfT) \cap BdS = \{f, cf\}$, and $fT \cap cfT = \{0\}$.

<u>Proof of 4</u>: Suppose $x \in eT \cap fT$. Then there exists $t \in T$ such that $x = et = ft$. Multiplying by a and b implies $aft = aet = et = ft$ and $bft = ft$; so $cft = abft = aft = ft = x$, and hence $x \in fT \cap cfT$. By part (3), $x = 0$ and $eT \cap fT = \{0\}$.

<u>Proof of 5</u>: For $x \in eT \cap cfT$, there exists $t \in T$ such that $x = et = cft$. Hence

$$x = et = aet = acft = bft = ft,$$

and $x \in eT \cap fT$ and by part (4) $x = 0$. Similarly, $fT \cap ceT = \{0\}$.

<u>Proof of 6</u>: It is clear now that the arcs, eT, ceT, fT, and cfT separate S into 4 regions, each region containing one and only one element of $H(1)$. Also, multiplication by elements of $H(1)$ maps these regions onto themselves in the natural way. Therefore (6) is obvious.

Proof of 7: Let $x \in \{s \in S: cs = s\}$, and since $cs = abs = x$, $bx = as$.

If $x \in Q$, then $ax \in aQ \cap bQ$, and hence $ax = 0$ and $x = 0$.

If $x \in aQ$, then $bx \in baQ = cQ$ and $ax \in a \cdot aQ = Q$ so $ax \in Q \cap cQ$, and hence $ax = 0$ and $x = 0$.

If $x \in bQ$, then $ax \in abQ = cQ$ and $bx \in b \circ bQ = Q$ so $ax \in cQ \cap Q$, and hence $ax = 0$ and $x = 0$.

If $x \in cQ$, then $ax \in acQ = bQ$ and $bx \in bcQ = aQ$ so $ac \in bQ \cap aQ$, and hence $ax = 0$ and $x = 0$.

Proof of 8, 9, and 10: By the proof of case one of Theorem 3.16, either $e \in E$ or $ce \in E$. If both are idempotent, then $e \in e^2$ and $ce = (ce)^2 = c^2e^2 = e$, a contradiction, since $ce \neq e$. Therefore, either $e \in E$ or $ce \in E$ but not both, and similarly, either $f \in E$ or $cf \in E$, but not both.

If $e \in E$, then $ceT \cap E = \{0\}$; for if $cet \in E$, then

$$cet = (cet)^2 = c^2e^2t^2 = et ,$$

and $cet \in ceT \cap eT = \{0\}$. Similarly if $ce \in E$, then $eT \cap E = \{0\}$. Note that the same statements hold if e is replaced by f.

Suppose $ce \in E$ and $cf \in E$; then $(eT \cup fT) \cap E = \{0\}$ and the arc $eT \cup fT$ separates 1 from cf and ce, or, 0 is a cut point of E, a contradiction. If $ce \in E$ and $f \in E$, then by the proof of Theorem 3.16, $fT \cup ceT \subseteq \overline{S \setminus E}$ and, therefore, either $E \subset bQ$ or $E \subseteq Q \cup aQ \cup cQ$. $E \not\subset bQ$ since $1 \not\in bQ$ and eT separates 1 from ce in $Q \cup aQ \cup cQ$, so $E \subset Q \cup aQ \cup cQ$, a contradiction.

By replacing e with f and f with e, it becomes clear that both cf and e can not be idempotent.

Now, after exhausting all other possible cases, e and f must be idempotent. Clearly $eT \cup fT = \mathrm{Bd}E \cap \overline{S \setminus E}$, and either $E \subset Q$ or $E \subseteq aQ \cup bQ \cup cQ$. Since cT separates e from f in $aQ \cup bQ \cup cQ$ and $cT \cap E = \{0\}$, $E \not\subset aQ \cup bQ \cup cQ$. Therefore, $E \subseteq Q$ and the proof of the theorem is complete.

<u>Corollary 3.18</u>. $ef = 0$.

<u>Proof</u>:

$$cef = abet = aebf = ef ,$$

so by part (7), $ef = 0$.

<u>Corollary 3.19</u>. If T and T_1 are min threads from 1 to 0, then $fT = fT_1$ and $eT = eT_1$.

Proof: From the theorem it is clear that

$$fT = \{s \in S : bs = s\} \cap E = fT_1 .$$

Corollary 3.20. If $x = aQ$ and T is any min thread from 1 to 0, then $xT \subset aQ$.

Proof: Suppose $xT \not\subset aQ$; then there exists $t \in T_1$ such that $xt_1 \in eT \cup cfT$. Assume $t = \sup\{t_1 \in T : xt_1 \in eT \cup cfT\}$ and let $h \in T$ such $h \leq t$. If $xt \in eT$, then $ext = xt$ and $exh = exth = eth = xh$. For $xt \in eT$, $x^2t = xt$ and it follows that $xh \in eT$. Hence $xT \subset aQ$. If $xt \in cfT$, then $bxt = xt$ and $bxh = bxth = xth = xh$, so $xh \in cfT \cup fT$. Now $xt = cft_1$ for some $t_1 \in T$, so $xh = xth = cft_1h = cf(t_1h) \in cfT$ and the corollary is established.

Note that aQ may be replaced by Q , bQ , or cQ and the corollary still holds.

Remark: It is clear from corollary 3.20 that if $xt \in \operatorname{int}(aQ)$ for $t \in T$, then $xT \subset aQ$.

Lemma 3.2. If B is the arc of BdS from e to f containing 1 and T is any min thread from 1 to 0, then $BT = Q$.

Proof: $BT \subseteq Q$ by corollary 3.20 and clearly the boundary of Q , $eT \cup fT \cup B$, is contained in BT. Now

T acts on BT by $m(bt,t_1) = btt_1$, and $m(bt,1) = bt$ and $m(bt,0) = 0$ for all $bt \in BT$, hence BT is acyclic and it follows that $BT = Q$.

We continue our investigation in this section by considering a class of semigroups where the semilattice structure of E has the following additional hypothesis: For $e \in E$ define $M(e) = \{f \in E: e \leq f\}$ and assume $M(e)$ is connected for all $e \in E$.

Semilattices on a 2-cell with this property have been studied in [4] with one of the results being:

Theorem 3.22. [4] Suppose E has a 1.

These are equivalent:

(i) For each e, $M(e)$ is a connected set;

(ii) The boundary of E is the union of two min threads from 1 to 0; and

(iii) E is the continuous homomorphic image of $I \times I$.

Throughout the remainder of this section, assume $M(e)$ is connected for each $e \in E$.

By Theorems 3.22 and 3.17, the arc of BdE from 1 to e is a min thread J, and the arc of BdE from 1 to f is a min thread K.

Lemma 3.23. $JK = E$

Proof: It is clear that $eK = eT$ and $fJ = fT_1$, so $BdE = J \cup K \cup eK \cup fJ \subset JK$. It follows that $(BdE)^2 \subset JK$ and by [4], $(BdE)^2 = E$. Hence $JK = E$.

Theorem 3.24. E is a continuous homomorphic image of $I \times I$ such that $(0,1)$ and $(1,0)$ are mapped onto e and f respectively.

Remark: The homomorphism of Theorem 3.22 [4] does not have the properties required by this Theorem.

Proof: Denote $I \times \{1\}$ by V and $\{1\} \times I$ by W and let α_1 and α_2 be iseomorphisms from V and W onto J and K respectively.

Define the map α from $I \times I$ onto E by

$$\alpha(a,b) = \alpha_1(a,1)\alpha_2(1,b) \quad .$$

For (a,b) and (c,d) in $I \times I$, $\alpha((a,b)(c,d)) = \alpha(\min\{a,c\}, \min\{b,d\}) = \alpha_1(\min\{a,c\}, 1)\alpha_2(1, \min\{b,d\})$ and $\alpha(a,b)\alpha(c,d) = \alpha_1(a,1)\alpha_2(1,b)\alpha_1(c,1)\alpha_2(1,d) = \min\{\alpha_1(a,1), \alpha_1(c,1)\}\cdot\min\{\alpha_2(1,b), \alpha_2(1,d)\}$. But since α_1 and α_2 are iseomorphisms, $\alpha_1(\min\{a,c\},1) = \min\{\alpha_1(a,1),\alpha_1(c,1)\}$ and $\alpha_2(1, \min\{b,d\}) = \min\{\alpha_2(1,b), \alpha_2(1,d)\}$, so it follows that α is a homomorphism.

Consider the following commuting diagram:

$$\begin{array}{ccc} V \times W & \xrightarrow{\alpha_1 \times \alpha_2} & J \times K \\ \Big\downarrow{\scriptstyle m_1} & & \Big\downarrow{\scriptstyle m_2} \\ I \times I & \xrightarrow{\alpha} & E \end{array}$$

The maps m_1 and m_2 are the corresponding multiplication functions. Now m_1 maps the compact set $V \times W$ onto $I \times I$, and $\alpha m_1 = m_2(\alpha_1 \times \alpha_2)$ is continuous, hence α is continuous. By lemma 3.23, m_2 is onto, hence α is onto and the proof is complete.

Remark: By [4] α is monotone and by [20] a simple closed curve (or arc) is mapped by α onto a simple closed curve (or arc), possibly degenerate. It follows from [7] that the monotone continuous homomorphism α maps a min thread onto a min thread or a single point.

Lemma 3.25. Let $\Delta = \{(a,b) \in I \times I : a = b\}$ and let $T = \alpha(A)$; then T is a min thread from 1 to 0 such that $T \setminus \{0,1\} \cap \mathrm{Bd}E = \square$.

Proof: Clearly T is a min thread from 1 to 0 [7]. Consider the simple closed curves $C_1 = (0,1)W \cup V \cup \Delta$ and $C_2 = (1,0)V \cup W \cup \Delta$. Since 1 and 0 are elements of both

$\alpha(C_1)$ and $\alpha(C_2)$, C_1 and C_2 are mapped onto simple closed curves.

Now $\alpha(V \cup (0,1)W) = J \cup eK$ and $\alpha(W \cup (1,0)V) = K \cup fJ$, so $\alpha(C_1) = cK \cup J \cup B_1$ and $\alpha(C_1) = fJ \cup K \cup B_2$ where B_1 and B_2 are arcs from 1 to 0 such that $B_1 \setminus \{0,1\} \cap (eK \cup J) = \square$ and $B_2 \setminus \{0.1\} \cap (fJ \cup K) = \square$. The min thread T is an arc from 1 to 0 contained in $eK \cup J \cup B_1$ and also contained in $fJ \cup K \cup B_2$. The simple closed curves $\alpha(C_1)$ and $\alpha(C_2)$ contain only two arcs each from 1 to 0, namely $eK \cup J$ and B_1 for $\alpha(C_1)$ and $fJ \cup K$ and B_2 for $\alpha(C_2)$. Since $\alpha(C_1) = \alpha(V \cup (0,1)W \cup \Delta) = eK \cup J \cup T$ and $\alpha(C_2) = fK \cup K \cup T$, it follows that $T = B_1 = B_2$ and $T \setminus \{0,1\} \cap \mathrm{Bd}E = \square$.

<u>Remark</u>: By corollary 3.19, $eK = eT$ and $fJ = fT$.

Let $x \in J \cup K \setminus \{e,f\}$ and $(a,b) \in V \cup W$ such that $\alpha(a,b) = x$. Note $(a,b) \notin (1,0)V \cup (0,1)W$. Consider the min thread $(a,b)\Delta$ from (a,b) to $(0,0)$. By the multiplication of $I \times I$, $(a,b)\Delta$ meets the boundary of $I \times I$ only at (a,b) and $(0,0)$.

<u>Lemma 3.26</u>. $\alpha((a,b)\Delta) = xT$ is a min thread from x to 0 such that $xT \cap \mathrm{Bd}E = \{x,0\}$.

<u>Proof</u>: Clearly xT is a min thread from x to 0,

and by using the same arguments of the preceeding lemma, it follows that $xT\setminus\{x,0\}\cap BdE = \Box$.

Lemma 3.27. If $x \in E\setminus eT \cup fT$, then $xT\setminus\{x,0\}\cap BdE = \Box$.

Proof: Let $(a,b) \in I\times I$ such that $a \neq 0$, $b \neq 0$ and $\alpha(a,b) = x$. If $a \leq b$, consider $(a,1)\Delta$; and if $b \leq a$, consider $(1,b)\Delta$. Without loss of generality, assume $a \leq b$. Now $(a,b)\Delta \subset (a,1)\Delta$ and hence $xT \subset \alpha(a,1)T$. But by Lemma 3.26, $\alpha(a,1)T\setminus\{\alpha(a,1),0\}\cap E = \Box$, so clearly $xT\setminus\{x,0\}\cap BdE = \Box$.

We now turn our attention to the structure of Q, remembering that B denotes the arc of BdQ from e to f.

Lemma 3.28. If $x \in B$ and $x^2 \in eT \cup fT$, then $x = x^2$ and $x \in E$.

Proof: For some $t \in T$, either $x^2 = et$ or $x^2 = ft$. Hence $x = xet$ or $x = xft$ and $ax = axet = xet = x$ or $bx = bxft = xft = x$. Now e is the only element of B left fixed by multiplication by a, and f is the only element of B left fixed by multiplication by b. Therefore, $x = e$ or $x = f$, and in either case $x^2 \leq x$ and $x = x^2$.

Theorem 3.29. $E = Q$.

Proof: It will suffice to show $B \subset E$ since $BT = Q$ by lemma 3.21.

For $x \in B$ consider xT where $T = \alpha(\Delta)$. The arc $cT \cup xT$ cuts S and separates e from f where $cT \cap E = \{0\}$ by the proof of Theorem 3.16. Therefore, for $h = \sup\{t \in T : xt \in E\}$, $xh \neq 0$. Clearly $xh \in BdE$, so $xh = (xh)^2 = x^2h$ and $x^2h \in x^2T \cap BdE$.

If $x^2 \in eT \cup fT$, then $x \in E$ by lemma 3.28. If $x^2 \notin eT \cup fT$, then $x^2T \backslash \{x^2, 0\} \cap BdE = \square$ by lemma 3,27. Therefore, $x^2h = x^2$ since $x^2h = xh \neq 0$ and multiplying by x yields $x = xh = x^2h = x^2$ and $x \in E$. Therefore $B \subset E$ and $E = Q$.

Theorem 3.30. S is the continuous monotone homomorphic image of $S_1 = I_1 \times I_1$, the semigroup defined in Example 1.

Proof: Express S_1 as $I \times I \cup (-1,1)(I \times I) \cup (1,-1)[I \times I \cup (-1,1)(I \times I)]$ and remember that S_2, the semigroup of Example 2 is $I \times I \cup (-1,1)(I \times I)$. That is, $S_1 = S_2 \cup (1,-1)S_2$. We now extend α, the map defined in Theorem 3.24, to S_2 in the obvious way. Consider the following commutative diagram

$$\begin{tikzcd} I\times I \arrow[r] \arrow[d, "f"'] & E \arrow[r, hook] \arrow[d, dash, "\rho_a"] & E\cup aE \arrow[d, "\rho_a"] \\ (-1,1)(I\times I) \arrow[r] & aE \arrow[r, hook] & E\cup aE \end{tikzcd}$$

The maps f and ρ_a are the homeomorphism of multiplication by $(-1,1)$ and a, respectively, and i is the injection map. β_1 is defined by $\beta_1(s,t) = \beta_1((-1,1)(-s,t)) = a\alpha(-s,t)$. Clearly β_1 is continuous and onto, and

$\beta_1|_{\{0\}\times I} = \alpha|_{\{0\}\times I}$ where $\{0\}\times I = I\times I\cap(-1,1)(I\times I)$.

Define the map β from S_1 onto $E\cup aE$ by

$$\beta(s,t) = \begin{cases} \alpha(s,t) & \text{if } (s,t)\in I\times I \\ \beta_1(s,t) & \text{if } (s,t)\in(-1,1)(I\times I). \end{cases}$$

It follows that β is continuous, and if (s,t) and (x,y) are elements of $I\times I$, then $\beta((s,t)(x,y) = \beta(s,t)\beta(x,y)$. If (s,t) and (x,y) are elements of $(-1,1)(I\times I)$, then $(s,t) = (-1,1)(-s,t)$ and $(x,y) = (-1,1)(-x,y)$, and $(-s,t)$ and $(-x,y)$ are elements of $I\times I$.

$$(s,t)(x,y) = (-s,t)(-x,y)$$

$$\beta((s,t)(x,y)) = \beta((-s,t)(-x,y)) = \beta(-s,t)\beta(-s,y)$$

$$= a\beta(-s,t)a\beta(-s,y) = \beta(s,t)\beta(x,y))\ .$$

If $(s,t) \in I \times I$ and $(x,y) \in (-1,1)(I \times I)$, then $(x,y) = (-1,1)(-x,y)$ where $(-x,y) \in I \times I$ and $\beta((s,t)(x,y)) = \beta((-1,1)(s,t)(x,y)) = a\alpha((s,t)(-x,y)) =$ $a\alpha(s,t)\alpha(-s,y) = \beta(s,t)\cdot a\alpha(-x,y) = \beta(s,t)\beta(x,y)$. Therefore, β is a homomorphism and it follows from the diagram that β_1 is monotone. Hence β is clearly monotone.

Next, consider the following commutative diagram.

$$\begin{tikzcd}
S_1 \arrow[r, "\beta"] \arrow[d, "f_1"'] & E \cup aE \arrow[r, hook, "i"] \arrow[d, dash, "\rho_b"] & S \arrow[d, "\rho_b"] \\
(1,-1)S_1 \arrow[r, "\gamma"] & bE \cup cE \arrow[r, hook, "i"] & S
\end{tikzcd}$$

The maps f_1 and ρ_b are the homeomorphisms of multiplication by $(1,-1)$ and b, respectively, and again i is the injection map. Define γ by $\gamma(s,t) = \gamma((1,-1)(s,-t)) = b\beta(s,-t)$.

By the same arguments as before, γ is an onto continuous monotone homomorphism.

Define the map Φ from S_1 onto S by

$$\Phi(s,t) = \begin{cases} \beta(s,t) & \text{if } (s,t) \in S_2 \\ \gamma(s,t) & \text{if } (s,t) \in (1,-1)S_2 \end{cases}$$

and if follows that Φ is the desired continuous monotone homomorphism.

We conclude this section with several questions.

Question 1. Without any additional hypothesis on E, is Q an inverse subsemigroup of S?

Question 2. What is a necessary and sufficient condition for $E = Q$?

Section II

Throughout this section, $H(1) \approx Z_2$, and a will denote the element of $H(1) \setminus \{1\}$ where the arcs A_1 and A_2 from 1 to a are the decomposition of BdS. By Theorem 3.16, this section is clearly divided into the following two cases:

(1) when $aA_1 = A_2$ and $aA_2 = A_1$ and

(2) when $aA_1 = A_1$ and $aA_2 = A_2$.

Case 1. Example 5 illustrates the existence of a semigroup of this type and indicates a procedure of how other examples can be derived. A characterization of this type is unknown at this time so we conclude case 1 with the following questions:

Question 3. Does there exist a semigroup of this type such that $E \cap \mathrm{Bd}S = \{1\}$?

Question 4. What is a necessary and sufficient condition for a semigroup of this type to be the continuous monotone homomorphic image of S_5?

Case 2. By the proof of Theorem 3.16, we can assume without loss of generality that the unique element e of A_1, such that $ae = e$, is idempotent. Let y denote the unique element of A_2 such that $ay = y$.

There are three natural classes of semigroups to consider in this case;

(i) when $y = 0$

(ii) when $y \in E \setminus \{0\}$ and

(iii) when $y \notin E$.

Remark: Examples 2,3, and 4 illustrate the existence of semigroups of each of the three classes.

Theorem 3.31. If $y = 0$, then

1) $eT \cap \mathrm{Bd}S = \{x, 0\}$

2) $eT - \mathrm{Bd}E \cap \overline{S \setminus E}$

3) If Q is the 2-cell bounded by eT and the arc B of $\mathrm{Bd}S$ from e to 0 containing 1, then $S = Q \cup aQ$ where $Q \cap aQ = eT$.

4) $E \subset Q$.

Proof: Since $eT = \{s \in S : as = s\}$, the theorem follows from Theorems 3.16, 3.17 and lemma 1.5.

Assume $M(f)$ is connected for each $f \in E$, and let J be the min thread of BdE from 1 to e, and let K be the min thread from 1 to 0 which does not contain e. The following lemmas and Theorem are stated without proof since their proofs are almost identical to the corresponding proofs of Lemmas 3.23, 3.25, 3.26, 3.27, 3.28 and Theorem 3.24 in Section I.

Lemma 3.32. $JK = E$.

Theorem 3.33. E is a continuous monotone homomorphic image of $I \times I$ such that $(0,1)$ is mapped onto e.

Lemma 3.34. Let $\Delta = \{(s,t) \in I \times I : s = t\}$ and let $T = \alpha(\Delta)$; then T is a min thread from 1 to 0 such that $T \setminus \{0,1\} \cap E = \square$.

Lemma 3.35. If $x \in J \cup K \setminus \{0,e\}$ and $(s,t) \in I \times I$ such that $\alpha(s,t) = x$, then $\alpha((s,t)\Delta) = xT$ is a min thread from x to 0 such that $xT \cap BdE = \{x,0\}$.

Lemma 3.36. If $x \in E \setminus eT$, then $xT \setminus \{0,x\} \cap BdE = \square$.

Lemma 3.37. If B denotes the arc of Q from e to 0 containing 1, $x \in B$, and $x^2 \in eT$, then $x = x^2$ and $x \in E$.

The parallel between this development and that of Section I ends, since Q does not necessarily equal E.

Example 8. Consider S_4, the semigroup of Example 4, and let $R = \{(0,b) \in S_4 : -1/2 \leq b \leq 1/2\}$, a closed ideal of S_4. Let S_8 be the Rees quotient modulo R [8]. This semigroup has the properties of case one class (i), $M(f)$ is connected for each $f \in E$, and E is a proper subset of Q.

Theorem 3.38. If $E = Q$, then S is the continuous monotone homomorphic image of $S_1 = I_1 \times I$, the semigroup defined in Example 2.

Proof: Extend the homomorphism of Theorem 3.33 in the obvious way, as in the proof of Theorem 3.30.

For classes (ii) and (iii) there are the obvious theorems similar to Theorem 3.31, which will be stated without proof.

Theorem 3.39. If $y \in E \setminus \{0\}$, then

1) $(eT \cup yT) \cap \mathrm{Bd}S = \{e, y\}$,
2) $eT \cup yT = \mathrm{Bd}E \cap \overline{S \setminus E}$,
3) If Q is the 2-cell bounded by $eT \cup yT$ and the arc B of $\mathrm{Bd}S$ from e to y containing 1, then $S = Q \cup aQ$,

where $Q \cap aQ = eT \cup yT$

4) $E \subset Q$.

Theorem 3.40. If $y \notin E$, then

1) $(eT \cup yT) \cap BdS = \{e, y\}$

2) $eT \subset BdE \cap \overline{S \setminus E}$

3) $eT \cup yT = \{s \in S : as = s\}$

4) If Q is the 2-cell bounded by $eT \cup yT$ and the arc B of BdS from e to y containing 1, then $S = Q \cup aQ$ where $Q \cap aQ = eT \cup yT$.

5) $E \subset Q$.

We conclude this section with the following example.

Example 9. Consider the subsemigroup of $S_3 = I_1 \times I_2$ defined by $S = \{(s,t) \in S_9 : t \geq -1/2\}$ and let the closed ideal R of S_9 be $\{(s,t) \in S_9 : s = 0 \text{ and } -1/2 \leq t \leq 1/2\}$. Let S_9 be the Rees quotient modulo R [8]. Properties of this semigroup are; 1) It is of case one class i) type, 2) $M(f)$ is not connected for all $f \in E$ and 3) $E = Q$. If e and f correspond to the points $(0,1)$ and $(1,-1/2)$ then $ef = 0$.

Conclusion. By combining Theorem 3.30 and lemma 1 of [4], we have the following result.

Theorem 3.41. If S is an inverse semigroup on a

2-cell with an identity 1 whose set of idempotents E has no cut point point and $H(1)\approx Z_2 \times Z_2$, then S is the continuous monotone homomorphic image of S_1 if and only if $M(e)$ is connected for each $e \in E$.

Example 11. Consider S_9 embedded in the plane on the rectangle $[-1,1]\times[0,1]$ where $(1,1)$, $(0,0)$, $(-1,1)$, $(0,1)$ and $(1,0)$ correspond to 1, 0, a, e and f and $a \in H(1)\setminus\{1\}$. Let $S_{10} = S_9 \cup \{(s,t) \in R^2 : (s_\partial -t) \in S_9\}$ or $S \cup (1,-1)S_9$, where multiplication of S_{10} is defined by

$$(s,t)\circ(x,y)=\begin{cases}(s,t)(x,y) \text{ if } (s,t),(x,y)\in S_9\\ (s_1-t)(x,-y) \text{ if } (s,t),(x,y)\notin S_9\\ (1,-1)[(s,t)(x,-y)] \text{ if } (s,t)\in S_9 \text{ and } (s,y)\notin S_9.\end{cases}$$

Properties of S_{10} are; (1) $H(1)\approx Z_2 \times Z_2$ (2) $E = \{(s,t) : s \geq 0 \text{ and } t \geq 0\}$, (3) $0 \in \text{int } S_{10}$, (4) $E = Q$ where Q is defined in Theorem 3.17 and (5) $M(e)$ is not connected for all $e \in E$.

It is clear from Examples 9 and 10 that if E is a semilattice on a 2-cell with an identity 1 and with min threads from e to 0 and f to 0 such that $ef = 0$, then examples of the type of Section I and Section II class (i) can be constructed.

<u>Question 5</u>. For Theorem 3.17, 3.31, 3.39 and 3.40, is Q necessarily a subsemigroup?

<u>Question 6</u>. If $M(e)$ is connected for each $e \in E$, must S be a continuous monotone homomorphic image of a subsemigroup of S_1?

CHAPTER IV

Introduction. In the field of algebraic inverse semigroups, idempotent-separating congruences and their relation to the semilattice of idempotents have been studied in [5], [6], [9], and [17]. The existence and characterization of a maximal idempotent-separating congruence μ is presented in [9]. In [17], a condition on the semilattice of idempotents is given which is sufficient to imply that the $\mathcal{H}$ equivalence is a congruence and hence $\mathcal{H} = \mu$

The purpose of this chapter is to show the existence of a lattice of idempotents for each element of the semigroup. These lattices are generated by the row and column idempotents of the powers of the elements. The original goal was to characterize μ in terms of these lattices. That is, if $a\mathcal{H}b$ and a and b generate the same lattice, then is $(a,b) \in \mu$? The answer to this question is no and an example is given. However, the condition that two $\mathcal{H}$-equivalent elements have the same lattice is shown to be stronger than the condition that the row idempotents of their powers are equal.

Preliminaries and Results. Throughout this chapter,

S will denote an algebraic inverse semigroup and E will denote it's semilattice of idempotents.

Definition 4.1. The row (column) idempotent of an element a is the unique idempotent e such that $a\mathcal{R}e$ $(a\mathcal{L}e)$.

For $a \in S$, the row idempotent is aa^{-1} and the column idempotent is $a^{-1}a$ [5].

Definition 4.2. A congruence ρ on S is called idempotent-separating if each ρ equivalence class contains at most one idempotent. The maximal idempotent-separating congruence μ is contained in $\mathcal{H}$ and characterized by [17];

$$(a,b) \in \mu \text{ if and only if } aea^{-1} = beb^{-1} \text{ for all } e \in E.$$

Theorem 4.3. The $\mathcal{H}$-equivalence is a congruence if and only if for any two idempotents e and f, $xe = ex$ for $x \in H(f)$.

Proof: If $\mathcal{H}$ is a congruence, then $\mathcal{H} = \mu$ and for $x \in H(f)$, $x\mu x^{-1}$ and $x^2\mu x^{-1}$. Hence $x^2ex^{-2} = x^{-1}ex$, and by multiplying by x, $x^2ex^{-2}x = x^{-1}ex^2$. But $x^{-1}x = xx^{-1} = f$, $x = fx = xf$ and $x^{-1} = x^{-1}f = fx^{-1}$ since $x \in H(f)$ so,

$$x^{-1}ex^2 = x^2ex^{-2}x = x^2ex^{-1}\cdot x^{-1}x = x^2ex^{-1}f = x^2ex^{-1}.$$

also, $x\mu x^{-1}$ implies $xex^{-1} = x^{-1}ex$ so,

$$xe = xf \cdot e = xef = xex^{-1} \cdot x = x^{-1}ex \cdot x = x^{-1}ex^2$$

$$= x^2ex^{-1} = x \cdot xex^{-1} = xx^{-1}ex = fex = efx = ex \ .$$

Conversely, for $a\mathscr{H}b$, $aa^{-1} = bb^{-1}$, $a^{-1}a = b^{-1}b$, ab^{-1} and ba^{-1} are elements of $H(aa^{-1})$, and $a^{-1}b$ and $b^{-1}a$ are elements of $H(a^{-1}a)$. Therefore, ab^{-1} , ba^{-1}, $a^{-1}b$, and $b^{-1}a$ commute with all idempotents. For any $e \in E$

$$aea^{-1} = ae \cdot e \cdot a^{-1}aa^{-1} = ae \cdot a^{-1}a \cdot ea^{-1} = aeb^{-1}b \cdot ea^{-1}aa^{-1}$$

$$= aeb^{-1}beb^{-1}ba^{-1} = aeb^{-1} \cdot beb^{-1} \cdot ba^{-1} = aeb^{-1} \cdot ba^{-1} \cdot beb^{-1}$$

$$= ae \cdot b^{-1}b \cdot a^{-1}b \cdot eb^{-1} = ae \cdot a^{-1} \cdot bb^{-1}b \cdot eb^{-1} = ae \cdot a^{-1}b \cdot eb^{-1}$$

$$= a \cdot e \cdot a^{-1}b \cdot eb^{-1} = aa^{-1}b \cdot e \cdot eb^{-1} = bb^{-1}beb^{-1} = beb^{-1} \quad .$$

Therefore, $(a,b) \in \mu$ and $\mu = \mathscr{H}$.

<u>Remark</u>: After this theorem had been established, it was found in a slightly different form as an exercise in [6].

<u>Lemma 4.4.</u> For $a \in S$, the row idempotent of a^n is $a^n a^{-n}$ and the column idempotent is $a^{-n}a^n$.

<u>Proof</u>: $aa^{-1}a = a$ and if $a^k a^{-k} a^k = a^k$, then $a^{-k}a^k \in E$ and

$$a^{k+1}\cdot a^{-(k+1)}\cdot a^{k+1} = a^k\cdot aa^{-1}\cdot a^{-k}a^k\cdot a = a^k a^{-k}a^k aa^{-1}a = a^{k+1}.$$

Therefore, $a^n = a^n a^{-n} a^n$ for all n, and similarly $a^{-n} = a^{-n}a^n a^{-n}$. Hence $a^n S \supset a^n a^{-n} S \supset a^n a^{-n} a^n S = a^n S$ and $a^n \mathcal{R} a^n a^{-n}$ and $Sa^n \supset Sa^{-n}a^n \supset Sa^n a^{-n} a^n = Sa^n$ and $a^n \mathcal{L} a^{-n} a^n$.

Lemma 4.5. For $a \in S$, let $e_n = a^n a^{-n}$ and $f_n = a^{-n}a^n$; then $e_n e_{n+r} = e_{n+r} = e_{n+r}$ for $r \geq 1$ and dually $f_n \cdot f_{n+r} = f_{n+r}$ for $r \geq 1$.

Proof:

$$a^n a^{-n}\cdot a^{n+r}\cdot a^{-(n+r)} = a^n a^{-n}\cdot a^n\cdot a^r\cdot a^{-(n+r)}$$

$$= a^n a^{-n} a^n\cdot a^r a^{-(n+r)} = a^n a^r\cdot a^{-(n+r)} = a^{n+r}\cdot a^{-(n+r)}.$$

Hence $e_{n+r} \leq e_n$ and $e_{n+r}e_n = e_{n+r}$. Similarly $f_n f_{n+r} = f_{n+r}$.

Lemma 4.6. If $e_n f_m \leq e_s f_t$, then $e_n f_m = e_{\max\{n,x\}}\cdot f_{\max\{m,s\}}$.

Proof:

$$e_n f_m = e_n f_m \cdot e_s f_t = e_n e_s f_m f_t = e_{\max\{n,s\}}\cdot f_{\max\{m,t\}}.$$

Lemma 4.7. If $e_n f_m = e_{n+1} f_m$, then

$e_n f_m = e_{n+r} f_m$ for all $r \geq 1$.

Proof: Multiply $e_n f_m = e_{n+1} f_m$ on the left by a and on the right by $a^{-m} a^{m-1}$.

$$a \cdot e_n f_m a^{-m} a^{m-1} = a e_{n+1} f_m a^{-m} a^{m-1}$$

$$a \cdot a^n a^{-n} \cdot a^{-m} a^m a^{-m} \cdot a^{m-1} = a a^{n+1} a^{-(n+1)} \cdot a^{-m} a^m a^{-m} \cdot a^{m-1}$$

$$a^{n+1} \cdot a^{-n} \cdot a^{-m} \cdot a^{m-1} = a^{n+2} a^{-(n+1)} a^{-m} \cdot a^{m-1}$$

$$a^{n+1} \cdot a^{-(n+1)} \cdot a^{-(m-1)} \cdot a^{m-1} = a^{n+2} \cdot a^{-(n+2)} \cdot a^{-(m-1)} \cdot a^{m-1}$$

Hence $e_{n+1} f_{m-1} = e_{n+2} f_{m-1}$ and multiplying by f_m implies $e_{n+1} f_m = e_{n+2} f_m$ and the lemma follows by induction.

Remark: Clearly if $e_n f_m = e_n f_{m+1}$, then $e_n f_m = e_n f_{m+r}$ for $r \geq 1$, by a dual argument.

Theorem 4.8. For $a \in S$

$\wedge_a = \{e_n f_m : n \geq 1 , m \geq 0$ where $e_n f_o = e_n$ for all $n\}$ is a lattice.

Proof: By lemma 4.5, $\wedge_a$ is closed under multiplication, and it is well known that a commutative idempotent semigroup is a semilattice where multiplication is the semilattice operation. Hence it will suffice to show the existence of a "join" or "cup" operation.

For elements $e_n f_m$ and $e_s f_t$ of Λ_a define

$$e_n f_m \vee e_s f_t = \begin{cases} e_n f_m & \text{if } e_s f_t \leq e_n f_m \\ e_s f_t & \text{if } e_n f_m \leq e_s f_t \\ e_p f_q & \text{otherwise, where } p = \min\{m,s\} \\ & \text{and } q = \min\{n,t\}. \end{cases}$$

If the two elements compare, it is clear that $e_n f_m \vee e_s f_t$ is the least upper bound. Hence assume that they do not compare and suppose $e_n f_m \leq e_i f_j$ and $e_s f_t \leq e_i f_j$. We now consider, all possible cases for i and j with respect to p and q.

<u>Case 1</u>. If $i \leq p$ and $j \leq q$, then $e_p \leq e_i$ and $f_q \leq f_j$ by lemma 4.5, and it follows that $e_p f_q \leq e_i f_j$.

<u>Case 2</u>. If $i > p$ and $j \leq q$, then

$$e_n f_m = e_n f_m \cdot e_i f_j = e_{\max\{n,i\}} \cdot f_m$$

$$e_s f_t = e_s f_t \cdot e_i f_j = e_{\max\{s,i\}} f_t .$$

If $i = \max\{n,i\} = \max\{s,i\}$, then $e_n f_m = e_i f_m$ and $e_s f_t = e_i f_t$, and it follows that $e_n f_m$ and $e_s f_t$ compare.

Since $i > p = \min\{n,s\}$, we may assume $i > n$ without loss of generality, so the only other case to consider is when $s = \max\{s,i\}$ and $s > i$. Therefore, $e_nf_m = e_if_m$ and by lemmas 4.5 and 4.7 $e_nf_m = e_sf_m$. Hence e_nf_m and e_sf_t compare, and the conditions of case 2 can not exist if e_nf_m and e_sf_t do not compare.

Case 3. If $i \leq p$ and $j > q$, then by a dual argument of case 2, e_nf_m and e_sf_t must compare, so case 3 does not exist if e_nf_m and e_sf_t do not compare.

Case 4. If $i > p$ and $j > q$, then

$$e_nf_m = e_nf_me_if_j = e_{\max\{i,n\}}f_{\max\{j,m\}}.$$

$$\text{and} \quad e_sf_t = e_sf_te_if_j = e_{\max\{i,s\}}\cdot f_{\max\{j,t\}}.$$

If $i = \max\{i,n\} = \max\{i,s\}$ or $j = \max\{j,m\} = \max\{j,t\}$, then by the argument of case 2, e_nf_m and e_sf_t compare. Hence we can assume that $s > i > n$ and consider the subcases when (a) $m > j > t$ or (b) $t > j > m$. If $m > j > t$, then $e_nf_m = e_if_m$, and by lemmas 4.5 and 4.7,

$$e_nf_m = e_if_m = e_sf_m,$$

and hence e_nf_m and e_sf_t compare. If $t > j > m$, then $e_nf_m = e_if_j$, so again e_nf_m and e_sf_t compare.

Therefore, if e_nf_m and e_sf_t do not compare and $e_i f_j$ is larger than or equal to both, then $i \leq p$ and $j \leq q$ and $e_p f_q \leq e_i f_j$.

Hence $e_p f_q$ is the less upper bound of the set of elements larger than or equal to $e_n f_m$ and $e_s f_t$, and Λ_a is a lattice.

Lemma 4.9. If $(a,b) \in \mu$, then $\Lambda_a = \Lambda_b$ and $\Lambda_a{-1} = \Lambda_b{-1}$.

Proof: It will suffice to show that $a^n a^{-n} = b^n b^{-n}$ and that $a^{-n}a^n = b^{-n}b^n$ for all $n \geq 1$.

For $(a,b) \in \mu$, it follows that $a \mathcal{H} b$, $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$, so

$$a^2a^{-2} = a(aa^{-1})a^{-1} = b(aa^{-1})b^{-1} = b(bb^{-1})b^{-1} = b^2b^{-2}$$

and by induction $a^n a^{-n} = b^n b^{-n}$ for all $n \geq 1$. μ is a congruence on S, so if $(a,b) \in \mu$, then $(a^{-1}, b^{-1}) \in u$ [17]. Therefore

$$a^{-2}a^2 = a^{-1}(a^{-1}a)a = b^{-1}(a^{-1}a)b = b^{-1}(b^{-1}b)b = b^{-2}b^2 ,$$

and again by induction $a^{-n}a^n = b^{-n}b^n$ for all $n \geq 1$.

The next lemma is obvious and is stated without a proof.

Lemma 4.10. If $\Lambda_a = \Lambda_b$, then the row idempotent of a^n is equal to the row idempotent of b^n for all $n \geq 1$.

Examples.

Definition 4.11 [5]. By a one-to-one partial transformation of a set X we mean a one-to-one mapping α of a subset Y of X onto a subset $Y_1 = Y\alpha$ of X. By the inverse α^{-1} of α we mean the mapping of $Y\alpha$ onto Y which is inverse to α in the usual sense of mappings. Let $\mathcal{I}_X$ denote the set of all one-to-one partial transformations of X, including that of the empty set $\square$ of X onto itself; this "empty transformation" will be denoted by 0. The product of two elements α and β of $\mathcal{I}_X$ is defined as follows. Let Y be the domain of α and Z that of β. If $Y\alpha \cap Z = \square$, define $\alpha\beta = 0$. Otherwise let $W = (Y\alpha \cap Z)\alpha^{-1}$ and define $\alpha\beta$ to be the iterate of $\alpha|_W$ and $\beta|_{W\alpha}$ in the usual sense. Clearly, $\alpha\beta$ is a one-to-one transformation of W onto $W\alpha\beta$, and so belongs to $\mathcal{I}_X$. Associativity is easily verified. Hence $\mathcal{I}_X$ is a semigroup, which is called the symmetric inverse semigroup on the set X.

If the set X is finite, then denote $\alpha \in \mathcal{I}_X$ by

$\binom{x_1, \ldots, x_n}{y_1, \ldots, y_n}$ where $\{x_1, \ldots, x_n\}$ is the domain of α, $\{y_1, \ldots, y_n\}$ is the range of α, and $\alpha(x_i) = y_i$.

Example 1. Let $X = \{a,b,c,d,e,f\}$,

$\alpha = \begin{pmatrix} a & b & c & d \\ a & e & b & f \end{pmatrix}$ and $\beta = \begin{pmatrix} a & b & c & d \\ a & f & b & e \end{pmatrix}$.

It follows that $\alpha^2 = \begin{pmatrix} a & c \\ a & e \end{pmatrix}$, $\alpha^n = \begin{pmatrix} a \\ a \end{pmatrix}$ for $n \geq 3$, $\beta^2 = \begin{pmatrix} a & c \\ a & f \end{pmatrix}$, and $\beta^n = \begin{pmatrix} a \\ a \end{pmatrix}$ for $n \geq 3$. Hence

$$\wedge_\alpha = \wedge_\beta$$

$$\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} a & c \\ a & c \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} a & b \\ a & b \end{pmatrix} \arrow[dl, dash] \\
 & \begin{pmatrix} a \\ a \end{pmatrix} &
\end{tikzcd}$$

But $\wedge_{\alpha^{-1}} \neq \wedge_{\beta^{-1}}$, since $\alpha^{-2}\alpha^2 = \begin{pmatrix} a & e \\ a & e \end{pmatrix}$ and $\beta^{-2}\beta^2 = \begin{pmatrix} a & f \\ a & f \end{pmatrix}$; therefore $(\alpha,\beta) \notin u$.

Example 2. Let $X = \{a,b,c,d,e,f,g,h\}$,

$\alpha \begin{pmatrix} a & b & c & d & e & f \\ a & c & e & f & g & h \end{pmatrix}$ and $\beta = \begin{pmatrix} a & b & c & d & e & f \\ a & c & f & e & g & h \end{pmatrix}$.

Now $\alpha^2 = \begin{pmatrix} a & b & c & d \\ a & e & g & h \end{pmatrix}$, $\alpha^3 = \begin{pmatrix} a & b \\ a & g \end{pmatrix}$, $\alpha^n = \begin{pmatrix} a \\ a \end{pmatrix}$ for $n \geq 4$,

$\beta^2 = \begin{pmatrix} a & b & c & d \\ a & f & h & g \end{pmatrix}$, $\beta^3 = \begin{pmatrix} a & b \\ a & h \end{pmatrix}$, and $\beta^n = \begin{pmatrix} a \\ a \end{pmatrix}$ for $n \geq 4$.

It follows that $\alpha^n\alpha^{-n} = \beta^n\beta^{-n}$ for all n, $\alpha\alpha^{-1}.\alpha^{-2}\alpha^2 = \begin{pmatrix} a & e \\ a & e \end{pmatrix}$, and $\beta\beta^{-1}\beta^{-2}\beta^2 = \begin{pmatrix} a & f \\ a & f \end{pmatrix}$. Therefore, the row idempotents of the powers of α and β are equal but $\Lambda_\alpha \neq \Lambda_\beta$.

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VITA

James Edward L'heureux was born August 5, 1934, in Edgefield, South Carolina. He attended the public schools in Greenville and Georgetown, South Carolina, graduating from high school in 1952. In the fall of that year he entered Louisiana State University, where he received his B.S. degree in Physics in the Spring of 1956. For the next three years he served as an officer in the United States Air Force. He re-entered L.S.U. in September, 1959 and received his M.S. degree in Mathematics at the Summer Commencement, 1961.

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Title of Thesis: Inverse Semigroups on the 2-cell

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July 24, 1969